The Valuation of Options on Bonds with Default Risk

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In this paper we present a model for valuing European and American options, which incorporates both default and interest rate risks. We develop a framework that permits evaluation of three kinds of options: (i) options issued by default-free counterparties on risky bonds, (ii) options issued by risky counterparties on default-free bonds and (iii) options issued by risky counterparties on risky bonds — a case where default risk enters at both levels. We show that the price of a put option on a risky discount bond is hump shaped for a European put and monotone increasing for an American put. We also find that the price impact of default risk is less for an American put option than for a European one (JEL: G13).

Keywords: option pricing, default risk, defaultable bonds, vulnerable options.

I. Introduction

In the finance literature, default risk was first modeled by Black and Scholes (1973) and Merton (1974), and then extended by Black and Cox (1976), Shimko, Tejima, and van Deventer (1993), Nielsen, Saá-Requejo, and Santa-Clara (1993), Longstaff and Schwartz (1995a), Briys and De Varenne (1997), and many others. These authors model corporate bonds and credit spreads. The purpose of this paper is to provide an efficient model, which can be used to price European and American options on bonds in the presence of default and interest rate risks. Three classes of options have been identified and analyzed. The first is when only the bond underlying the option is subject to default risk. The second is when only the issuer of the option is subject to

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default, and the third is when default risk enters at both levels, which is the case of options issued by a risky writer on risky bonds.

Since the early 1990's, the pricing of options with default risk is a topic that has been discussed in increasing detail within finance literature. Johnson and Stulz (1987) were the first to develop a model for pricing European options issued by a risky writer, called vulnerable options. Their assumption is that interest rates are constant and default occurs when the value of the option exceeds the value of the assets of the counterparty upon exercise. Hull and White (1995) have extended the Johnson and Stulz (1987) model when default can occur at any time prior to maturity of the option. They postulate that default occurs if the value of the assets of the option writer falls below a specified default boundary, and that only a fraction of the nominal amount of a claim is paid out in case of default. However, their work does not consider the situation where the underlying assets are subject to default risk.

Das (1995) has presented a valuation method for options on the credit risk of a corporate debt. He has developed a pricing formula where interest rates are assumed to be constant and where the dynamics of interest rates are derived from the term structure model of Heath, Jarrow and Morton (1992). Das (1995) has not studied vulnerable options. Jarrow and Turnbull (1995) have used a foreign currency analogy for pricing vulnerable options and options on risky bonds. They assume that the two processes — generating the interest rate and the bankruptcy, respectively — are independent.

This paper is organized as follows: Section II provides the notations and presents the valuation framework. Section III presents the valuation model and simulation results for options on risky bonds. Section IV presents the valuation model for vulnerable options on risk-free bonds, and section V derives a valuation model and simulation results when two sources of default risk are present - the case of vulnerable options on risky bonds. Conclusions are made in section VI.

II. The Valuation Framework

Assumption 1. We assume perfect and frictionless markets. Trading takes place continuously. There are no taxes nor transaction costs.

Assumption 2. Let $S_0(t)$ denote the riskless security. Its dynamics are given by:

$$\frac{dS_0}{S_0} = r(t)\,dt,$$

where r denotes the short-term riskless interest rate. The dynamics of r are given by:

$$dr = a(b-r)\,dt + \sigma_r \sqrt{r}\,dz_r\,, \tag{1}$$

where a, b, and σ_r are constants and z_r is a standard Wiener process. The assumption regarding the dynamics of r is derived from the term structure model of Cox, Ingersoll and Ross (1985). As it does not allow negative rates, this model fits the dynamics of interest rate better than the Ornstein-Uhlenbeck process.

Assumption 3. Let V be the total value of the assets of the firm. The dynamics of V are given by:

$$dV = (\mu - d)Vdt + \sigma_v V dz_v\,, \tag{2}$$

where μ and σ_v are constants and z_v is a standard Wiener process. The processes z_r and z_v are correlated and the instantaneous correlation between dz_r and dz_v is ρdt. We allow for a constant payout rate to the investors of the firm, d. We assume that the value of the firm does not depend on the capital structure of the firm. This is the standard assumption that the Modigliani-Miller Theorem holds. In the modelling approach adopted in this paper, we consider a firm whose capital structure may include a variety of debts with different maturity dates and coupon rates.

Assumption 4. Default occurs at the first date at which the value of the firm's assets fall below a critical level. This critical level is defined as the total value D of the debt faced by the firm. We assume that the process followed by D is:

$$\frac{dD(r(t),t)}{D(r(t),t)} = [r(t) - \delta]\,dt, \tag{3}$$

where δ is a constant payout rate to the debtholders of the firm. It is usually assumed that default is triggered when the value of the assets of a firm falls below a constant default barrier, e.g. Longstaff and Schwartz (1995a), and Kim, Ramaswamy, and Sundaresan (1993). This assumption presents some degree of inconsistency for it implies that the default barrier should be the same for all debts of the firm whatever their maturity dates are.

In practice, and as reported by Saá-Requejo and Santa-Clara (1999), default occurs when the value of the firm falls below the value of the liabilities (stock-based insolvency) or when the firm fails to make a cash payment (flow-based insolvency). Our default barrier is defined as the face value of the liabilities. Hence, the drift of $D(r(t),t)$ depends on time and on the riskless interest rate.

Assumption 5. In case of default only a fraction $(1 - \eta)$ of the no-default value of the security is paid at default date. Although, the percentage writedown on a security in case of default depends on its seniority, we shall assume for simplicity that η is a constant.

Under the risk neutral probability measure, the dynamics of r and V are given by:

$$dr = [a(b-r) - \lambda\sigma_r r]dt + \sigma_r \sqrt{r}\, dw_r \,, \qquad (4)$$

$$dV = (r-d)Vdt + \sigma_v V dw_v, \qquad (5)$$

where λ is a constant and $\lambda\sqrt{r}$ is the market price of the interest rate risk. w_r et w_v are two standard Wiener processes under the risk neutral probability and ρdt is the instantaneous correlation between dw_r and dw_v.

Let $\overline{B}(t,r,V)$ denote value at time t of a defaultable discount bond contingent on the value of V and r and promising $M + C$ dollars at date $T_b \geq t$. Using assumptions 1, 2 and 3 as well as equations (4) and (5), the value of the defaultable discount bond must satisfy the following fundamental partial differential equation:

$$\begin{aligned}&\overline{B}_t + (r-d)V\overline{B}_V + [a(b-r) - \lambda\sigma_r r]\overline{B}_r + \\ &\frac{\sigma_r^2}{2} r\overline{B}_{rr} + \frac{\sigma_v^2}{2}V^2\overline{B}_{VV} + \rho\sigma_v\sigma_r\sqrt{r}V\overline{B}_{Vr} - r\overline{B} = 0\end{aligned} \qquad (6)$$

The subscripts denote first and second order partial derivatives with respect to r, V and t. The value of the defaultable discount bond is obtained by solving equation (6) subject to the appropriate boundary conditions. At maturity, the bondholder would receive the face value of the bond if default did not occur during its life:

$$\overline{B}(T_b, r, V) = M + C. \tag{7}$$

We propose to solve equation (6) using modified version of the explicit finite difference method suggested by Hull and White (1990). Brennan and Schwartz (1978) have shown that the explicit finite difference method is equivalent to a trinomial lattice approach. They have also shown how a transformation of variables ensures convergence when stock options are being valued. Hull and White (1990) have modified the explicit finite difference method and have used both a transformation of variables and a new branching process.

As Brennan and Schwartz (1978) and Hull and White (1990) demonstrate, it is appropriate to make a transformation of variables to obtain constant instantaneous standard deviation. From equations (4) and (5), the appropriate transformations of r and V are:

$$X = \sqrt{r} \tag{8}$$

$$Z = Ln(V). \tag{9}$$

From Ito's lemma, the processes followed by X and Z under the risk neutral probability measure are:

$$dX = A(X)\,dt + g\,dw_r \tag{10}$$

where $$g = \frac{\sigma_r}{2},$$

$$A(X) = \frac{4ab - \sigma_r^2}{8X} - \frac{X}{2}(a + \lambda\sigma_r),$$

and $$dZ = \left(r - d - \frac{\sigma_v^2}{2} \right) dt + \sigma_v dw_v. \tag{11}$$

The next stage is to transform variables again in order to eliminate the correlation between dw_r and dw_v. Let Y denote the new variable defined as follows:

$$Y = Z - \alpha X \text{ where } \alpha = 2\rho \frac{\sigma_v}{\sigma_r}. \tag{12}$$

From Ito's lemma, the process followed by Y under the risk neutral probability measure is:

$$dY = H(X)\,dt + f d\overline{w}_v,$$

where $$H(X) = \left(X^2 - d - \frac{\sigma_v^2}{2} \right) - \alpha A(X),$$

and $$f = \sigma_v \sqrt{1-\rho^2}\,. \tag{13}$$

$\overline{w}_v$ is a new Wiener process, uncorrelated with w_r and defined as follows: $$dw_v - \rho dw_r = \sqrt{1-\rho^2}\, d\overline{w}_v. \tag{14}$$

To implement the explicit finite difference method, we divide the time interval $[t_0, T_b]$ into n subintervals $[t_i, t_{i+1}]$ of equal length, where $t_i = t_0 + i\Delta t$, $\Delta t = \frac{T_b - t_0}{n}$, $i = 1,\ldots,n$. We denote $X_0 + j\Delta X$ and $Y_0 + k\Delta Y$ respectively, by X_j and Y_k. The relationship between ΔX, ΔY and Δt is:[1]

$$\Delta X = g\sqrt{3\Delta t} \text{ and } \Delta Y = f\sqrt{3\Delta t}$$

The partition of the (t,X,Y) space forms a grid. Let (i, j, k) denote the node of the three dimensional grid representing the point of coordinates $t_i = t_0 + i\Delta t$, $X_0 + j\Delta X$ and $Y_0 + k\Delta Y$, with $i = 1,\ldots, n$ and $j,k = -i,\ldots, i$. Brennan and Schwartz (1978) and Hull and White (1990) show that the explicit finite difference method is equivalent to a trinomial lattice approach. For instance, X_j can reach three possible values $X_j + \Delta X$, X_j and $X_j - \Delta X$ at time T_{i+1}. The probabilities of moving from X_j to X_{j+1}, X_j and X_{j-1} are chosen to respect the first and second moments of the change in X in the time interval Δt , and to make the sum equal to unity:[2]

1. See Hull and White (1990, p. 94).

2. The Δt^2 terms are ignored.

$$q_{j,\,j+1} + q_{j,\,j} + q_{j,\,j-1} = 1,$$

$$q_{j,\,j+1}\,(\Delta X) + q_{j,\,j-1}\,(-\Delta X) = E[X(t+\Delta t) - X(t)],$$

$$q_{j,\,j+1}(\Delta X)^2 + q_{j,\,j-1}(-\Delta X)^2 = E[(X(t+\Delta t) - X(t))^2],$$

where $q_{j,\,j+1}$, $q_{j,\,j}$ and $q_{j,\,j-1}$ are the probabilities of moving from X_j to X_{j+1}, X_j and X_{j-1} respectively. It is possible to verify that this discrete time process converges towards the underlying continuous-time process as $\Delta t \to 0$. It can be clearly expressed that:

$$q_{j,\,j+1} = \frac{1}{6} + \frac{A(X)^2\,\Delta t^2}{2\Delta X^2} + \frac{A(X)\,\Delta t}{2\Delta X}\,,$$

$$q_{j,\,j} = \frac{2}{3} - \frac{A(X)^2\,\Delta t^2}{\Delta X^2}\,,$$

$$q_{j,\,j-1} = \frac{1}{6} + \frac{A(X)^2\,\Delta t^2}{2\Delta X^2} - \frac{A(X)\,\Delta t}{2\Delta X}\,.$$

Since X can take any positive value, $A(X)$ is not bounded. It follows that, in some situations, the standard explicit finite difference method may not ensure positive probabilities and convergence. A modified method should then be used, as described in Hull and White (1990), whom have proposed alternative branching procedures to ensure that the probabilities associated with all three branches remain positive. It can be shown that the constraint of positivity implies that $X_{\min} \le X \le X_{\max}$, where:[3]

$$X_{\min} = \frac{-\beta + \sqrt{\beta^2 + 4e_1e_2}}{2e_2},$$

$$X_{\max} = \frac{\beta + \sqrt{\beta^2 + 4e_1e_2}}{2e_2},$$

3. See Hull and White (1990, p. 95).

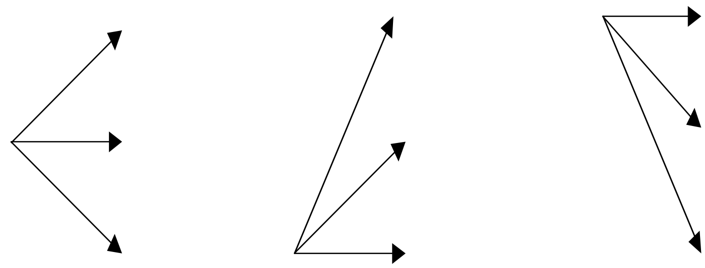

FIGURE 1.— Alternative Branching Procedures. (a) is the standard branching procedure, (b) is the procedure when X_{min} is reached and (c) is the procedure when X_{max} is reached.

and
$$\beta = \frac{\Delta X}{2\Delta t}.$$

Figure 1 shows three possible branches in the tree. When the value $X_{\min}$ is reached, a new branching procedure is used (figure 1(b)) and the three possible values that could be reached by X_j after Δt are X_j, $X_j + \Delta X$ and $X_j + 2\Delta X$. When the value $X_{\max}$ is reached, the branching procedure used is described in figure 1(c) and the three possible values that could be reached by X_j after Δt are X_j, $X_j - \Delta X$ and $X_j - 2\Delta X$. The branching probabilities in these cases are obtained in the same way as they are in the case of figure 1(a).

When X_{max} is reached:

$$q_{i,j} = \frac{7}{6} + \frac{A(X)^2 \Delta t^2}{2\Delta X^2} + \frac{3A(X)\Delta t}{2\Delta X}$$

$$q_{j,j-1} = -\frac{1}{3} - \frac{A(X)^2 \Delta t^2}{\Delta X^2} - 2\frac{A(X)\Delta t}{\Delta X}$$

$$q_{j,j-2} = \frac{1}{6} + \frac{A(X)^2 \Delta t^2}{2\Delta X^2} + \frac{A(X)\Delta t}{2\Delta X}$$

When X_{min} is reached:

$$q_{j,j+2} = \frac{1}{6} + \frac{A(X)^2 \Delta t^2}{2\Delta X^2} - \frac{A(X)\Delta t}{2\Delta X}$$

$$q_{j,j+1} = -\frac{1}{3} - \frac{A(X)^2 \Delta t^2}{\Delta X^2} + 2\frac{A(X)\Delta t}{\Delta X}$$

$$q_{j,j} = \frac{7}{6} + \frac{A(X)^2 \Delta t^2}{2\Delta X^2} - \frac{3A(X)\Delta t}{2\Delta X}$$

Since $H(X)$ depends only on X, $H(X)$ is bounded. It follows that, for the variable Y, the branching procedure used corresponds to figure 1(a). At each node of the trinomial lattice, Y_k can reach three possible values $Y_k + \Delta Y$, Y_k and $Y_k - \Delta Y$ in the time interval Δt. It is easy to verify that the branching probabilities in this case. They are:

$$p_{k,k+1} = \frac{1}{6} + \frac{H(X)^2 \Delta t^2}{2\Delta Y^2} + \frac{H(X)\Delta t}{2\Delta Y},$$

$$p_{k,k} = \frac{2}{3} - \frac{H(X)^2 \Delta t^2}{\Delta Y^2},$$

$$p_{k,k-1} = \frac{1}{6} + \frac{H(X)^2 \Delta t^2}{2\Delta Y^2} - \frac{H(X)\Delta t}{2\Delta Y},$$

where $p_{k,k+1}$, $p_{k,k}$ and $p_{k,k-1}$ are the probabilities of moving Y_k to Y_{k+1}, Y_k and Y_{k-1} respectively.

The objective of this paper is to apply modified version of the explicit finite difference method, as presented above, to the valuation of European and American options on bonds in presence of default risk. Three types of options need to be distinguished: (i) options issued by default-free counterparties on risky bonds, (ii) options issued by risky counterparties on default-free bonds and (iii) options issued by risky counterparties on risky bonds, a case where the default risk enters at both levels.

III. Pricing Options on Defaultable Bonds

In this section, we apply the modified explicit finite difference method presented above to the valuation of options on risky bonds, where only the issuers of the bonds are subject to default risk. Let (t, X, Y) denote the three dimensional space where $t \in [t_0, T_b]$, t_0 is the current time, T_b is the maturity of the risky bond, and $T_p \leq T_b$ is the maturity of a European or an American put with strike price K on the risky bond.

A. Valuing Options on Risky Discount Bonds

For simplicity we consider the case of options on a risky discount bond. Let $\overline{B}_{i,j,k}$ be the value of a risky zero-coupon bond at node (i, j, k) = (t_i, X_j, Y_k). According to assumptions 4 and 5, default occurs at the first date at which the value of firm's assets falls below the nominal value of the debts faced by the firm, and in case of default only a constant fraction of the no-default value of the bond is paid at default date. At maturity, the value of the risky discount bond is:

$$\overline{B}_{n,j,k} = \left\{ \begin{array}{lll} M + C & \text{if} & D(t_n) \leq V(t_n) \\ (1-\eta)(M+C) & \text{if} & D(t_n) > V(t_n) \end{array} \right\}, \tag{15}$$

where
$$V(t_n) = \exp(Y_{n,k} + \alpha X_{n,j}), \tag{16}$$

$$D(t_n) = D(t_0)\exp((X^2_{n,j} - \delta)n\Delta t) \text{ for all } j, k = -n, \ldots\ldots, n, \tag{17}$$

and η is the percentage writedown on the risky discount bond in the case of default. The value of the risky discount bond prior to maturity can be calculated using the risk-neutral valuation. The value of $\overline{B}$ at time T is known. The value of $\overline{B}$ at time t_i is equal to the expected value at time t_{i+1}, in a risk neutral world, discounted at time t_i using the risk-free interest rate. Since the probabilities obtained in section II are risk-neutral, the price $\overline{B}_{i,j,k}$ at node (i, j, k) of a risky discount bond is obtained from its prices at all nodes in the next period through the backward equation:

$$\overline{B}_{i,j,k} = \begin{cases} E_{Q_{t_i}}\left[\overline{B}_{i+1,j,k}\right]e^{-X_j^2(t_i)\Delta t} & \text{if} \quad D(t_i) \le V(t_i) \\ (1-\eta)E_{Q_{t_i}}\left[\overline{B}_{i+1,j,k}\right]e^{-X_j^2(t_i)\Delta t} & \text{if} \quad D(t_i) > V(t_i) \end{cases},$$

where

$$E_{Q_{t_i}}\left[\overline{B}_{i+1,j,k}\right] = \sum_{\theta=-1}^{1}\sum_{\zeta=-1}^{1} q_{j,j+\theta}\, p_{k,k+\zeta}\, \overline{B}_{i+1,j+\theta,k+\zeta},$$

for all

$$i = n-1, \ldots\ldots, 0 \text{ and } j,k = -i, \ldots\ldots, i, \tag{18}$$

and $E_{Q_{t_i}}$ denotes expectations under the risk neutral probability, Q.

Let $\overline{PE}_{i,j,k}$ and $\overline{PA}_{i,j,k}$ be the values of a European and an American put options written on a risky discount bond at node (i, j, k). At maturity T_p, the values of these puts are:

$$\overline{PE}_{m,j,k} = \overline{PA}_{m,j,k} = \max\left[K - \overline{B}_{m,j,k};0\right],$$

where
$$m = \frac{T_p - t_0}{\Delta t}, \tag{19}$$

and $j, k = -m, \ldots\ldots, m.$

Before maturity, the values of these puts are obtained by backward equations in the same way as they are for $\overline{B}_{i,j,k}$:

$$\overline{PE}_{i,j,k} = E_{Q_{t_i}}\left(\overline{PE}_{i+1,j,k}\right)e^{-X_j^2(t_i)\Delta t}, \tag{20}$$

$$\overline{PA}_{i,j,k} = \max\left[E_{Q_{t_i}}\left(\overline{PA}_{i+1,j,k}\right)e^{-X_j^2(t_i)\Delta t}; K - \overline{B}_{i,j,k}\right], \tag{21}$$

where $i = m-1, \ldots\ldots, 0$ and $j, k = -i, \ldots\ldots, i.$

Simulated results for put options on risky discount bonds are reported in table 1, table 2 and figure 2. The parameter values used are

TABLE 1. Prices of European and American Put Options on Risky Discount Bond for Different Values of *D*/*V*.

D/V	T_p	European Put Value	American Put Value
	1	0.5495	0.6346
	2	0.8072	0.8963
0.40	3	1.0512	1.1757
	4	1.2026	1.5126
	5	0.4643	1.9455
	1	2.2071	2.3146
	2	3.0239	3.2403
0.50	3	3.5214	4.0919
	4	3.6641	5.0324
	5	1.2397	6.0723
	1	4.8042	4.9317
	2	6.3190	6.9924
0.60	3	6.9397	8.8191
	4	6.9301	10.6416
	5	1.8082	12.2362

Note: The parameter values used are $T_b = 5$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $d = 0.05$, $\sigma_v = 0.2$, $\rho = -0.25$, $r = 0.0318$, $\eta = 0.5$, $\Delta t = 0.05$, $\delta = 0.08$, $M = 100$, $C = 5$ and $K = \overline{B}(0,5)$.

$K = \overline{B}(0,5)$, $M = 100$, $C = 5$, $r = 0.0318$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $\sigma_v = 0.2$, $d = 0.05$, $\delta = 0.08$, $\eta = 0.5$ and $\Delta t = 0.05$. In our model, the factor η represents the percentage writedown on a security in case of default. On a sample of defaulted bond issues during the 1985 to 1991 period, Altman (1992) reports average writedown rates ranging from 39% to 80% depending on the seniority of the issue. Values of a, b and σ_r are based on empirical tests of the Cox, Ingersoll and Ross model made by Belhaj (2002) on french data. The other parameters correspond to a standard calibration.

The effects of capital structure are reported in table 1. Capital structure is reflected by the ratio D/V. In table 1, we vary the ratio D/V from 0.4 to 0.6. Several interesting observations can be made. First, and as expected, the value of a put option on a risky discount bond is an increasing function of D/V. When the ratio D/V increases, the price of a risky discount bond decreases, and thus the value of the put increases. Second, the value of the American put is higher than that of the

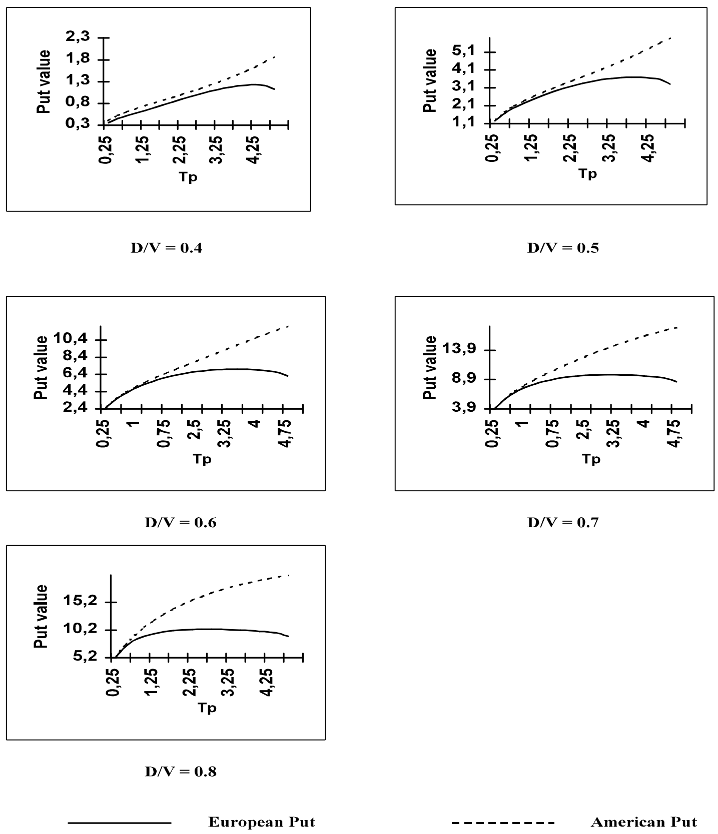

FIGURE 2.— Values of European and American Put Options on Risky Discount Bond for Different Values of D/V. The parameter values used are $T_b = 5$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $d = 0.05$, $\sigma_v = 0.2$, $\rho = -0.25$, $r = 0.0318$, $\eta = 0.5$, $\Delta t = 0.05$, $\delta = 0.08$, $M = 100$, $C = 5$ and $K = \overline{B}(0,5)$.

European put. The reason for this is because the holder of an American put on a risky discount bond, can decide to exercise it early in order to alleviate the impact of the default risk.

Table 1 and figure 2 show the effects of put maturity T_p when the risky discount bond's maturity is fixed at 5 years. One can see that the value of the American put is a monotonic increasing function of T_p.

TABLE 2. Prices of European and American Put Options on Risky Discount Bond for Different Values of η and σ_v.

η	σ_v	Credit Spreads (Basis Points)	European Put Value	American Put Value
	0.15	36.63	0.4170	0.4745
0.25	0.20	163.57	2.7169	2.9197
	0.25	379.16	5.7509	6.2211
	0.15	41.14	0.4956	0.5601
0.50	0.20	182.31	3.0239	3.2403
	0.25	417.16	6.1810	6.6870
	0.15	43.24	0.5327	0.6002
0.75	0.20	190.77	3.1601	3.3828
	0.25	434.10	6.3630	6.8877

Note: The parameter values used are $T_b = 5$, $T_p = 2$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $d = 0.05$, $\rho = -0.25$, $r = 0.0318$, $D/V = 0.5$, $\Delta t = 0.05$, $\delta = 0.08$, $M = 100$, $C = 5$ and $K = \overline{B}(0,5)$.

However, when $D/V = 0.4$, the maximum value for the European put occurs with a maturity of about 4.25 years. However, when $D/V = 0.8$, the maximum value for the European put occurs with a maturity of about 2.75 years. This is because the value of a European put on the risky discount bond contains both the time value from the put option and the time value from the risky discount bond, which is also an option on firm value. The time value of the risky discount bond decreases when both D/V increases and $T_p \to T_b$.

Table 2 shows the effects of the percentage writedown η on the risky discount bound in case of default and the volatility σ_v on European and American put values. When η increases, the credit spread increases and the European and American put values increase as a result. For instance, when $\sigma_v = 0.2$, as η increases from 0.25 to 0.75, the European and American put values increase from 2.7169 and 2.9197 to 3.1601 and 3.3828, respectively. Consequently, the percentage increase on the price is significantly less for an American put than it is for a European put.

It is also apparent from table 2 that if σ_v increases, then there is a tendency for $\overline{B}$ to decrease and for the value of the put option to increase. When $\eta = 0.5$ the credit spread increases from 41.14 to 417.16 basis points as σ_v increases from 0.15 to 0.25 and the values of European and American put options increase from 0.4956 and 0.5601 to 6.1810 and 6.6870, respectively. Thus, the impact of default risk on the price of American put option is less important than it is on the price of a European put option.

B. Valuing Options on Risky Coupon Bonds

In this section, we extend the discrete time model presented in the previous subsection to value options on risky coupon bonds. Longstaff and Schwartz (1995a) and Cathcart and El-Jahel (1998) developed a model in which risky coupon bonds can be valued as simple portfolios of discount bonds. This decomposition presents some inconsistency in the sense that the default barrier is assumed to be constant (independent from the maturity of the discount bond) and the same for all discount bonds. Geske (1977) used the contingent claim approach to value coupon bonds with a constant interest rate. He states, therefore, that a corporate coupon bond can be evaluated as a compound option.

Let $\overline{B}_{i,j,k}$ be the value at node $(i, j, k) = (t_i, X_j, Y_k)$ of a risky coupon bond promising N coupons. At maturity, the value of the risky coupon bond is obtained in the same way as for a discount bond by equation (15). At time t_i, the value of risky coupon bond under the risk neutral probability is:

$$\overline{B}_{i,j,k} = \left[C_i + E_{Q_{t_i}} \left(\overline{B}_{i+1,j,k} \right) e^{-X_j^2(t_i)\Delta t} \right] \left(1 - \eta I_{\{D(t_i) > V(t_i)\}} \right), \qquad (22)$$

where C_i is the coupon paid at the end of period i, $i = N - 1, \ldots\ldots, 0$, and $j, k = -i, \ldots\ldots, i$. Given the value of the risky coupon bond at each node (i, j, k) of the three-dimensional lattice, the values of European and American put options are obtained from equations (19), (20) and (21).

C. Valuing Options on Risky Convertible Bonds

An advantage of the discrete time approach adopted in this paper is that it can be easily extended to value options on risky convertible bonds. In this subsection, we examine the effects of default risk on options written on a risky convertible bonds. A convertible bond is a hybrid bond which allows its bearer to exchange it for a given number of shares of stock anytime before the maturity date of the bond. As a consequence, a convertible bond is equivalent to a portfolio of two securities: a non-convertible bond with the same coupon rate and maturity as the convertible bond, and a call option written on the stock of the firm. The early conversion feature makes the convertible bond similar to an American option. Brennan and Schwartz (1977), Kim, Ramaswamy, and Sundaresan (1993), and Tsiveriotis and Fernandes (1998) have analyzed the valuation of corporate convertible bonds, but their work does not examine the case of options on these bonds.

In this subsection, we apply the modified version of the explicit finite difference method to the valuation of options written on a corporate convertible bonds. We consider a convertible bond, $\overline{B}^{c}$, maturing at time T_b, convertible at any time to shares of the underlying stock, and paying at expiration $M + C$ if not converted. At maturity date, the value of the risky convertible bond is:[4]

$$\overline{B}^{c}_{n,j,k} = \begin{cases} \Omega(V(t_n) - D(t_n)) & \text{if} \quad V(t_n) - D(t_n) \geq \dfrac{M+C}{\Omega} \\ M + C & \text{if} \quad 0 \leq V(t_n) - D(t_n) < \dfrac{M+C}{\Omega} \\ (1-\eta)(M+C) & \text{if} \quad \text{default} \end{cases} \quad (23)$$

where Ω is the conversion ratio and indicates the number of underlying stock shares into which the bond may be converted. In the same way as for a non-convertible bond, the value of a risky convertible bond before maturity is obtained using backward equations in relation to conversion and default conditions at each node of the branching tree. Thus, the values of European and American options written on this bond are obtained from equations (19), (20) and (21).

Figure 3 plots the value of a European put option written on a convertible bond with and without default risk as a function of the value of the firm. We assume that the put option is at-the-money and the conversion ratio is equal to 0.5. As the value of the firm goes to infinity, the default probability decreases and thus the value of a put option on the risky convertible bond converges towards the value of a put option on a default-free convertible bond. In the case where the value of the firm falls, it will not be optimal to convert the bond and, as shown in the

4. I assume that the convertible bond is non callable by the issuer and non putable for a cash amount by the holder.

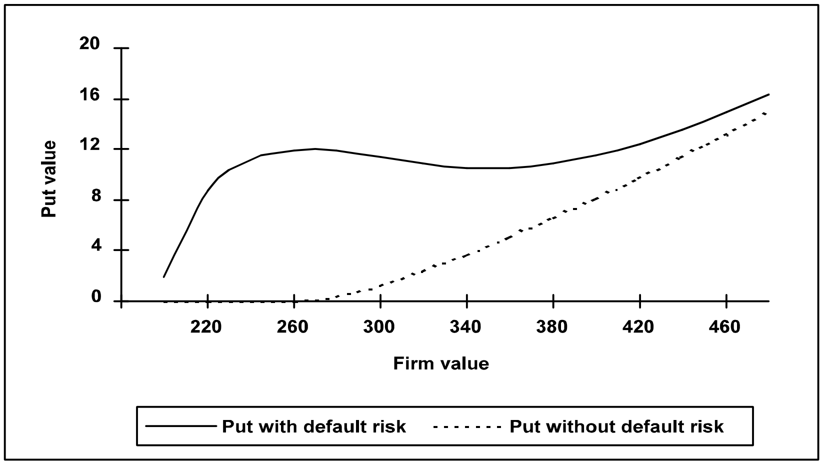

FIGURE 3.— Values of European Put Option on Convertible Bond with and without Default Risk. The parameter values used are $T_b = 5$, $T_p = 2$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $d = 0.05$, $\sigma_v = 0.2$, $\rho = -0.25$, $r = 0.0318$, $\eta = 0.5$, $\Delta t = 0.05$, $D = 200$, $\delta = 0.08$, $M = 100$, $C = 5$, $\Omega = 0.5$ and $K = \overline{B}(0,5)$.

previous section, the value of a put option is an increasing function of ratio D/V. Such is not the case for a put option on a default free convertible bond.

IV. Pricing Vulnerable Options

When an option is priced, it is usually assumed that there is no risk that the counterparty writing the option will default. But as the over-the-counter market develops, the default risk of the option writer becomes very significant. Johnson and Stulz (1987) have derived pricing formulas for vulnerable European options where the option has been assumed to be the sole liability of the counterparty.

In this section, we examine the impact of default risk of the option writer on European and American options' prices. For convenience, we assume that the asset underlying the option is a default-free discount bond. Let $B(t, r)$ denote value at time t of a default-free discount bond

TABLE 3. Percentage Reduction Arising from Default Risk in the Prices of Vulnerable European and American Call Options.

T_c	European Call	American Call
Panel A: Call Options are at-the-money K = B(0,5) = 89.4743		
0.5	0.00	0.00
1.0	0.02	0.01
1.5	0.20	0.11
2.0	0.74	0.38
2.5	1.63	0.82
3.0	2.82	1.39
3.5	4.22	2.07
4.0	5.77	2.83
4.5	7.44	3.63
5.0	9.14	4.47
Panel B: Call Options are out-of-the-money K = 95, B(0,5) = 89.4743		
0.5	0.00	0.00
1.0	0.00	0.00
1.5	0.00	0.00
2.0	0.38	0.35
2.5	1.45	1.33
3.0	2.70	2.26
3.5	4.14	3.22
4.0	5.72	4.22
4.5	7.41	5.23
5.0	9.14	6.25

Note: The parameter values used are $T_b = 5$, $D/V = 0.5$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $d = 0.05$, $\sigma_v = 0.2$, $\eta = 0.5$, $\rho = -0.25$, $r = 0.0318$, $\Delta t = 0.05$, $\delta = 0.08$, $M = 100$, and $C = 5$.

promising $M + C$ dollars at date T_b and CE^v and CA^v be the values of European and American call options written on a risk-free discount bond, respectively, with strike price K and expiry date $T_c \leq T_b$. In the present case, V and D designate the value of the assets and the value of the debts of the firm of the option writer, respectively. Applying the valuation model presented in section II, the values of European and American call options at date T_c are:

$$CE^v_{m,j,k} = CA^v_{m,j,k} = \max\left[B_{m,j} - K;0\right]\left(1-\eta I_{\{D(t_m)>V(t_m)\}}\right),$$

$$m = \frac{T_c - t_0}{\Delta t},$$

$$j,\, k = -m, \ldots\ldots\ldots, m.$$

$V(t_m)$ and $D(t_m)$ are obtained from equations (16) and (17), respectively. At node (i, j, k), the values of call options derive from the following backward equations:

$$CE^{v}_{i,j,k} = E_{Q_{t_i}}\left[CE^{v}_{i+1,j,k}\right] e^{-X_j^2(t_i)\Delta t}\left(1 - \eta I_{\{D(t_i) > V(t_i)\}}\right), \qquad (24)$$

$$CA^{v}_{i,j,k} = \max\left(E_{Q_{t_i}}\left[CA^{v}_{i+1,j,k}\right] e^{-X_j^2(t_i)\Delta t}; B_{i,j} - K\right)\left(1 - \eta I_{\{D(t_i) > V(t_i)\}}\right), \qquad (25)$$

$$i = m - 1 \ldots\ldots\ldots, 0,$$
$$j,\, k = -i, \ldots\ldots\ldots, i.$$

Table 3 shows the effect of the option maturity T_c on the vulnerable European and American call options prices. In panel A and panel B, the option maturity is varied from 0.5 year to 5 years, the bond maturity being fixed at 5 years. It appears that the percentage reduction in the option price increases with option maturity ; this is because the default risk of the option writer itself increases with respect to maturity. For example, as shown in panel A of table 3, the percentage reduction in the price of a vulnerable European call at-the-money is equal to 2.82 percent when the option time to maturity is fixed at 3 years and is equal to 9.14 percent when the option time to maturity is fixed at 5 years.

It is also apparent from table 3 that the percentage reduction in vulnerable option prices is less for an American option than for a European one. As noted in subsection III A, the reason for this is that the American option holder can decide to exercise early in order to eliminate the impact of default risk.[5] For instance, when the option time to maturity is fixed at 4 years, the percentage reduction in the price of a vulnerable European option at-the-money is equal to 5.77 percent and it is equal to 2.83 percent for a vulnerable American option.

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5. The holder of the vulnerable American option is fully informed about V and D and he decides to exercise when D is just below V. This is not the case for a default-free American option.

TABLE 4. Percentage Reduction Arising from Default Risk in the Prices of Vulnerable European and American Call Options.

		D/V		
r (%)		0.30	0.50	0.70
2.18	European Call	0.00	0.72	15.73
	American Call	0.00	0.48	12.28
3.18	European Call	0.00	0.38	11.07
	American Call	0.00	0.35	10.95
4.18	European Call	0.00	0.00	0.00
	American Call	0.00	0.00	0.00

Note: The parameter values used are $T_b = 5$, $T_c = 2$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $d = 0.05$, $\sigma_v = 0.2$, $\eta = 0.5$, $\rho = -0.25$, $\Delta t = 0.05$, $\delta = 0.08$, $M = 100$, $C = 5$, and $K = 95$.

Table 4 provides results for the percentage reduction in the prices of vulnerable European and American options for different values of r and D/V. As the default risk increases with the ratio D/V, the percentage reduction in the vulnerable option prices is an increasing function of D/V. It can also be seen from table 4 that the vulnerable American option is less sensitive to default risk than the European one.

Table 4 also shows that the percentage of price reduction for a vulnerable call option decreases as the riskless interest rate increases. This is explained by the fact that when the interest rate tends to infinity, both the default-free and vulnerable options are worthless and the percentage reduction should tend to zero.

V. Pricing Vulnerable Options on Risky Bonds

In this section we examine the case of options issued by a risky writer on risky bonds, a case where two sources of default risk are present. Let V_1 be the total value of the assets of the bond issuer, and V_2 be the total value of the assets of the option writer. According to assumption 3, the dynamics of V_1 and V_2 under the risk neutral probability are given by:

$$dV_1 = (r - d_1)V_1 dt + \sigma_{v_1} V_1 dw_{v_1} ,$$

$$dV_2 = (r - d_2)V_2 dt + \sigma_{v_2} V_2 dw_{v_2} \ ,$$

where $dw_{v_1} dw_r = \rho_{r,v_1} dt, dw_{v_2} dw_r = \rho_{r,v_2} dt$ and $dw_{v_1} dw_{v_2} = \rho_{v_1,v_2}$. In the same way as when there is only one source of default risk, the appropriate transformations of V_1 and V_2 are:

$$Y_1 = 1\text{n}(V_1) - \alpha_1 X$$

$$Y_2 = 1\text{n}(V_2) - \alpha_2 X.$$

From Ito's lemma, the processes followed by Y_1 and Y_2 are:

$$dY_1 = H_1(X)dt + f_1 d\overline{w}_{v_1},$$

$$H_1(X) = \left(X^2 - d_1 - \frac{\sigma_{v_1}^2}{2} \right) - \alpha_1 A(X),$$

$$f_1 = \sigma_{v_1} \sqrt{1 - \rho_{r,v_1}^2},$$

$$\alpha_1 = 2\rho_{r,v_1} \frac{\sigma_{v_1}}{\sigma_r},$$

where $\overline{w}_{v_1}$ is a new Wiener process, uncorrelated with w_r and defined as follows:

$$dw_{v_1} - \rho_{r,v_1} dw_r = \sqrt{1 - \rho_{r,v_1}^2}\, d\overline{w}_{v_1},$$

$$dY_2 = H_2(X)dt + f_2 d\overline{w}_{v_2},$$

with

$$H_2(X) = \left(X^2 - d_2 - \frac{\sigma_{v_2}^2}{2} \right) - \alpha_2 A(X),$$

$$f_2 = \sigma_{v_2} \sqrt{1 - \rho_{r.v_2}^2},$$

$$\alpha_2 = 2\rho_{r,v_2} \frac{\sigma_{v_2}}{\sigma_r},$$

where $\overline{w}_{v_2}$ is a new Wiener process, uncorrelated with w_r and defined as follows:

$$dw_{v_2} - \rho_{r,v_2} dw_r = \sqrt{1-\rho^2_{r,v_2}}\, d\overline{w}_{v_2}.$$

It is also appropriate to make another transformation in order to offset correlation between $\overline{w}_{v_1}$ et $\overline{w}_{v_2}$. Let Y_3 denote a new variable such that:

$$Y_3 = Y_2 - \alpha_3 Y_1 ,$$

$$\alpha_3 = \rho_{v_1,v_2} \frac{f_2}{f_1}.$$

The process followed by Y_3 is:

$$dY_3 = H_3(X)dt + f_3 d\overline{\overline{w}}_{v_2}, \tag{26}$$

with $$H_3(X) = \left(X^2 - d_2 - \frac{\sigma^2_{v_2}}{2} \right) - \alpha_2 A(X) - \alpha_3 H_1(X),$$

$$f_3 = f_2\sqrt{1-\rho^2_{v_1,v_2}} = \sigma_{v_2}\sqrt{1-\rho^2_{r,v_2}}\sqrt{1-\rho^2_{v_1,v_2}},$$

where $\overline{\overline{w}}_{v_2}$ is a new Wiener process, uncorrelated with $\overline{w}_{v_1}$ and defined as follows:

$$d\overline{w}_{v_2} - \rho_{v_1,v_2} d\overline{w}_{v_1} = \sqrt{1-\rho^2_{v_1,v_2}}\, d\overline{\overline{w}}_{v_2}.$$

The probabilities, $l_{k,\,k-1}$, $l_{k,\,k}$ et $l_{k,\,k+1}$, of moving from Y_{3j} to Y_{3j-1},Y_{3j} and Y_{3j+1} are obtained in the same way as for X et Y_2.

$$l_{k,k+1} = \frac{1}{6} + \frac{H_3(X)^2 \Delta t^2}{2\Delta Y_3^2} + \frac{H_3(X)\Delta t}{2\Delta Y_3},$$

$$l_{k,k} = \frac{2}{3} - \frac{H_3(X)^2 \Delta t^2}{\Delta Y_3^2},$$

$$l_{k,k-1} = \frac{1}{6} + \frac{H_3(X)^2 \Delta t^2}{2\Delta Y_3^2} - \frac{H_3(X)\Delta t}{2\Delta Y_3}$$

Let $\overline{PE}^v$ et $\overline{PA}^v$ be the values for vulnerable European and American put options written on a risky discount bond, respectively, with strike price K and expiry date T_p. The terminal values of the puts at date T_p are:

$$\overline{PE}^v_{m,j,k} = \overline{PA}^v_{m,j,k} = \max\left[K - \overline{B}_{m,j,k};0\right]\left(1 - \eta_2 I_{\{D_2(t_m) > V_2(t_m)\}}\right),$$

where for all $j,k = -m,\ldots\ldots,m$,

$$V_1(t_m) = \exp((Y_1)_{m,k} + \alpha_1 X_{m,j}),$$

$$V_2(t_m) = \exp((Y_3)_{m,k} + \alpha_2 X_{m,j} + \alpha_3 (Y_1)_{m,k}),$$

$$D_1(t_m) = D_1(t_0)\exp((X^2_{m,j} - \delta_1)(t_m - t_0)),$$

$$D_2(t_m) = D_2(t_0)\exp((X^2_{m,j} - \delta_2)(t_m - t_0)),$$

$$m = \frac{T_p - t_0}{\Delta t}.$$

η_1 and η_2 are the percentage writedowns on the risky discount bond and on the vulnerable option in case of default, respectively. At date t_i, the values of the puts are obtained from the backward equation:

$$\overline{PE}^v_{i,j,k} = E_{Q_{t_i}}\left[\overline{PE}^v_{i+1,j,k}\right] e^{-X_j^2(t_i)\Delta t}\left(1 - \eta_2 I_{\{D_2(t_i) > V_2(t_i)\}}\right),$$

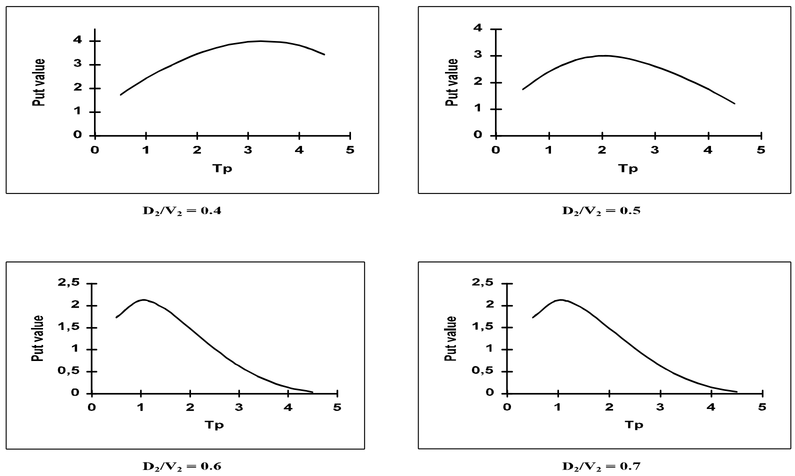

FIGURE 4.— Value of Vulnerable European Put Option on Risky Discount Bond for Different Values of D_2/V_2. The parameter values used are $T_b = 5$, $D_1/V_1 = 0.50$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, λ= 1%, $d_1 = d_2 = 0.05$, $\sigma_{v_1} = \sigma_{v_2} = 0.2$, $\rho_{r,v_1} = \rho_{r,v_2} = \rho_{v_1,v_2} = -0.25$, $r = 0.0318$, $\eta_1 = \eta_2 = 0.5$, $\Delta t = 0.05$, $\delta_1 = \delta_2 = 0.08$, $M = 100$, $C = 5$ and $K = \overline{B}(0,5)$.

$$\overline{PA}^{v}_{i,j,k} = \max\left(E_{Q_{t_i}}\left[\overline{PA}^{v}_{i+1,j,k}\right]e^{-X_j^2(t_i)\Delta t}; K - \overline{B}_{i,j,k}\right)\left(1-\eta_2 I_{\{D_2(t_i)>V_2(t_i)\}}\right),$$

$$i = m-1, \ldots\ldots\ldots, 0,$$
$$j, k = -i, \ldots\ldots\ldots, i,$$

where:

$$\overline{B}_{i,j,k} = E_{Q_{t_i}}\left[\overline{B}_{i+1,j,k}\right]e^{-X_j^2(t_i)\Delta t}\left(1-\eta_1 I_{\{D_1(t_i)>V_1(t_i)\}}\right),$$

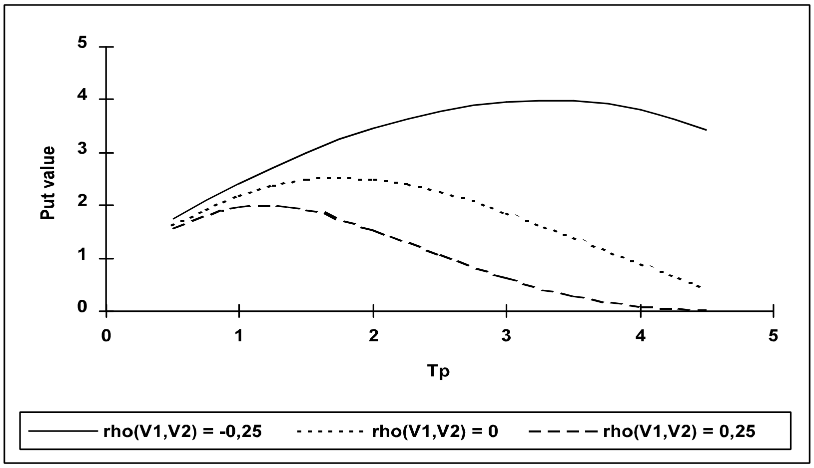

FIGURE 5.— Value of Vulnerable European Put Option on Risky Discount Bond for Different Values of $\rho_{v1,v2}$. The parameter values used are $T_b = 5$, $D_1/V_1 = D_2/V_2 = 0.50$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $d_1 = d_2 = 0.05$, $\sigma_{v_1} = \sigma_{v_2} = 0.2$, $\rho_{r,v_1} = \rho_{r,v_2} = -0.25$, $r = 0.0318$, $\eta_1 = \eta_2 = 0.5$, $\Delta t = 0.05$, $\delta_1 = \delta_2 = 0.08$, $M = 100$, $C = 5$ and $K = \overline{B}\,(0,5)$.

$$E_{Q_{t_i}}\left[\overline{B}_{i+1,j,k}\right] = \sum_{\theta=-1}^{1}\sum_{\zeta=-1}^{1} q_{j,j+\theta}\, p_{k,k+\zeta}\, \overline{B}_{i+1,j+\theta,k+\zeta},$$

$$E_{Q_{t_i}}\left[\overline{PE}^{v}_{i+1,j,k}\right] = \sum_{\theta=-1}^{1}\sum_{\zeta=-1}^{1} q_{j,j+\theta}\, l_{k,k+\zeta}\, \overline{PE}_{i+1,j+\theta,k+\zeta},$$

$$E_{Q_{t_i}}\left[\overline{PA}^{v}_{i+1,j,k}\right] = \sum_{\theta=-1}^{1}\sum_{\zeta=-1}^{1} q_{j,j+\theta}\, l_{k,k+\zeta}\, \overline{PA}_{i+1,j+\theta,k+\zeta}.$$

Simulated results are presented in table 5 and figures 4, 5 and 6. Figure 4 plots the value of a vulnerable European put option on a risky discount bond for different values of ratio D_2/V_2. As shown, the maximum values of the put occur at different maturities as ratio D_2/V_2 increases. For instance, when $D_2/V_2 = 0.4$, the maximum put value occurs for an option

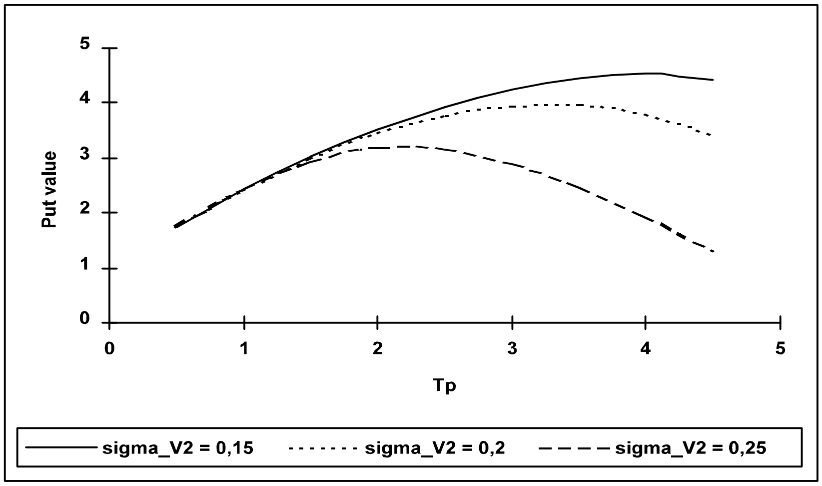

FIGURE 6.— Value of Vulnerable European Put Option on Risky Discount Bond for Different Values of σ_{v2}. The parameter values used are $T_b = 5$, $D_1/V_1 = D_2/V_2 = 0.50$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $d_1 = d_2 = 0.05$, $\sigma_{v_1} = 0.2$, $\rho_{r,v_1} = \rho_{r,v_2} = \rho_{v_1,v_2} = -0.25$, $r = 0.0318$, $\eta_1 = \eta_2 = 0.5$, $\Delta t = 0.05$, $\delta_1 = \delta_2 = 0.08$, $M = 100$, $C = 5$ and $K = \overline{B}\,(0,5)$.

with a maturity of about 4 years. When $D_2/V_2 = 0.7$, however, the maximum occurs for an option with a maturity of about 1 year. This is because for a lower ratio D_2/V_2 the put value is essentially influenced by the default risk of the underlying asset. In this case, the put value increases with respect to option time-to-maturity. However, for a higher ratio D_2/V_2 the put value is essentially influenced by the default risk of the option writer. In this case, the put value decreases as $T_p \to T_b$.

In figures 5 and 6 we plot the values of a vulnerable European put option on a risky discount bond for different values of ρ_{v_1,v_2} and σ_{v_2}. The graph shows that the put value decreases when the correlation ρ_{v_1,v_2} increases. This is due to the fact that ρ_{v_1,v_2} affects the dynamics of the option writer's asset value. In particular, an increase in ρ_{v_1,v_2} has a negative effect on the downward drift of the risk-neutral process for Y_3. This means, that as ρ_{v_1,v_2} increases, the default risk of the option writer

TABLE 5. Prices of Vulnerable European and American Put Options on Risky Discount Bond for Different Values of D_1 and D_2.

		D_2			
D_1		160	200	240	280
160	European Put	1.2901	1.0768	0.3851	0.0163
	American Put	1.3912	1.3912	1.3829	1.1348
200	European Put	4.1819	3.9605	2.6079	0.6309
	American Put	4.7556	4.7556	4.7517	4.5007
240	European Put	8.0228	7.8249	6.4288	2.8686
	American Put	10.0505	10.0505	10.0505	9.8004
240	European Put	10.9774	10.8155	9.6356	5.9598
	American Put	15.5839	15.5839	15.5839	15.5834

Note: The parameter values used are $T_b = 5$, $V_1 = V_2 = 400$, $a = 0.03578$, $b = 0.03435$, $\sigma_r = 0.00725$, $\lambda = 1\%$, $d_1 = d_2 = 0.05$, $\sigma_{v_1} = \sigma_{v_2} = 0.2$, $\rho_{r,v_1} = \rho_{r,v_2} = \rho_{v_1,v_2} = -0.25$, $r = 0.0318$, $\eta_1 = \eta_2 = 0.5$, $\Delta t = 0.05$, $\delta_1 = \delta_2 = 0.08$, $M = 100$, $C = 5$ and $K = \overline{B}(0,5)$.

increases. In the same way, as shown by figure 6, an increase in σ_{v_2} implies an increase in default risk of the option writer and the put value decreases as a result.

Table 5 shows the prices of vulnerable European and American put options on a risky discount bond for different values of D_1 and D_2. The results show that the American put value is less sensitive to changes in D_2. As discussed above, this is because the holder of the option can reduce the impact of default risk by an early exercise. However, as D_1 increases, both the European and the American put values increase. For example, when $D_2 = 200$, as D_1 increases from 160 to 280, the European and the American put values increase from 1.0768 and 1.3912 to 10.8155 and 15.5839, respectively. This result also holds for non-vulnerable European and American put options on a risky discount bond (the results are shown in figure 2 and discussed in section III).

VI. Conclusion

In this article, we have developed a model for valuing European and American options on bonds, which incorporates both default and interest rate risks. An important feature of our model is that it can be applied to value different kinds of options: options issued by default-free counterparties on risky bonds, options issued by risky counterparties on default-free bonds, and options issued by risky counterparties on risky bonds — a case where default risk enters at both levels. It is assumed that default risk occurs when the value of the firm falls below a critical level, which depends on default-free interest rate and the time left to maturity. In addition, our approach allows defaulting to occur at any time prior to the maturity of the option and, in case of default, only a constant fraction is paid. The results show that the price of a put option on a risky bond is hump-shaped for a European put, and monotone increasing for an American put. Another important result is that the impact of default risk on the price of an American put option is less important than it is on the price of a European one. The discrete time approach adopted in this article can easily be extended to value other derivatives with path-dependent features.

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