Multi-Fractality in Foreign Currency Markets

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Several empirical studies have shown the inadequacy of the standard Brownian motion (sBm) as a model of asset returns. To correct for this evidence some authors have conjectured that asset returns may be independently and identically Pareto-Lévy stable (PLs) distributed, whereas others have asserted that asset returns may be identically - but not independently - fractional Brownian motion (fBm) distributed with Hurst exponents, in both cases, that differ from 0.5. In this article we empirically explore such non-standard assumptions for both spot and (nearby) futures returns for five foreign currencies: the British Pound, the Canadian Dollar, the German Mark, the Swiss Franc, and the Japanese Yen.

Keywords: exponent of Hurst, fractional Brownian motion, multi-fractal market hypothesis, Pareto-Lévy stable process, R/S analysis.

I. Introduction

The standard hypothesis concerning the behavior of asset returns in financial markets claims that they are independently and identically lognormally distributed ($\ln[P(t + dt) - \ln[P(t)] \sim N(\mu\, dt, \sigma^2 dt)$). The

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corresponding underlying stochastic process is characterized by a quantity, called the Hurst exponent H, which is related to some fractal aspects of the process itself.[1] In particular, for a standard Brownian motion (sBm) the Hurst exponent is $H = 0.5$.

Several empirical studies have supported the independent and identical lognormal behavior of asset returns, but others have shown its inadequacy as a model of asset returns. This inadequacy is often caused by the existence of many outliers, nonstationarity in the variance level, presence of asymmetry, and short and long-term dependence. Authors such as Lo and MacKinlay (1988), Lo (1991), Peters (1991, 1994), Evertsz (1995a, 1995b), Evertsz and Berkner (1995), Corazza (1996), Campbell, Lo and MacKinlay (1997) and Corazza, Malliaris and Nardelli (1997) provide statistical evidence that asset prices do not follow random walks.

To account for this discrepancy, some authors have conjectured that financial returns may be independently and identically Pareto-Lévy stable (PLs) distributed, whereas others have conjectured that asset returns may be identically, but not independently, fractional Brownian motion (fBm) distributed.[2,3] Both of these conjectures are characterized by exponents of Hurst such that $H \neq 0.5$.

In this article we consider such non-standard hypotheses about returns for both spot and (nearby) futures for five foreign currency markets: the British Pound, the Canadian Dollar, the German Mark, the Swiss Franc and the Japanese Yen. We assume the Hurst exponent H belongs to a suitable neighborhood of 0.5, that is, we (indirectly) assume that the stochastic process generating exchange rate returns can be either a PLs or a fBm motion. This assumption provides a more flexible theoretical framework to examine if the so-called Fractal Market

1. There is extensive literature on fractality from a mathematical point of view, such as Mandelbrot and Van Ness (1968) and Falconer (1990). Applications of fractality in finance are presented in Evertsz (1995a, 1995b), Evertsz and Berkner (1995), and Corazza, Malliaris and Nardelli (1997).

2. See Mittnik and Rachev (1993) and Campbell, Lo and MacKinlay (1997).

3. Representative references include Lo (1991), Peters (1991, 1994), Corazza (1996), Evertsz (1995a, 1995b), Evertsz and Berkner (1995), Belkacem, Levy Vehel, and Walter (1996), Ostasiewicz (1996), Campbell, Lo and MacKinlay (1997) and Corazza, Malliaris and Nardelli (1997).

Hypothesis (FMH), as proposed in Peters (1991, 1994), is a reasonable generalization of the standard Efficient Market Hypothesis (EMH), initially elaborated in Fama (1970). Of course, when $H = 0.5$ the FMH coincides with the EMH. Furthermore, we also assume that the Hurst exponent is a function of time, $H = H(t)$, allowing the foreign currency markets structures to vary over time. The introduction of this dynamic dimension permits the generalization of the FMH into the MultiFractal Market Hypothesis (MFMH).

Briefly, the MFMH provides a theoretical framework to account for changes from "regular" to "irregular" phases of the capital markets and vice versa. In general, in such markets traders have investment horizons with similar or different lengths. If the matching between the asset demand and supply is relatively equal, then both the liquidity and regularity of the markets are ensured, otherwise the opposite holds.[4, 5] Of course, when $H = 0.5$, for all suitable t, then the MFMH coincides with the EMH.

The remainder of the article is organized as follows: in section 2 we give a brief review of the literature; in sections 3 and 4 we present some theoretical and empirical aspects that are essential to our analysis; section 5 describes the data and section 6 reports the results of the multifractal analysis. In section 7 we offer an economic interpretation of our results, and finally, in section 8**,** we summarize our concluding remarks.

II. Review of the Literature

Market efficiency has been the most celebrated theory of financial markets during the past three decades. In its simplest formulation this theory claims that changes in asset prices reflect fully and instantaneously the release of all new relevant information. Furthermore, because such a flow of information cannot be anticipated between the current trading period and the next one, asset price changes, in efficient markets, are serially independent. In other words, the release of

4. Notice that the peculiarities of such a matching depends on the stochastic process generating the asset returns.

5. These concepts are discussed in detail in Pancham (1994), Corazza (1996), Belkacem, Vehel, and Walter (1996) and also in section 6 of this article.

unanticipated information moves asset prices randomly. The textbook by Campbell, Lo and MacKinlay (1997, section 8) explains various versions of the random walk hypothesis.

The efficient market theory, from its earliest formulation by Samuelson (1965) and Fama (1970), has been refined in several directions. Analytically, the concept of information has been rigorously defined. Statistically, the notion of random walk has been generalized to Itô processes. Moreover, the efficient market hypothesis has been extensively tested. Fama (1991) traces the evolution of the market efficiency theory during its first two decades and skillfully cites numerous studies that offer empirical support as well as empirical rejection of the EMH.

In this article we conduct an empirical investigation of the return behavior of five foreign currencies in order to detect possible discrepancies between the actual behavior of such currencies and the classical random walk. Note that we do not claim that foreign currency markets are inefficient nor do we assert that the EMH does not hold. We acknowledge that market efficiency is currently the central theory of financial economics, at least until a new theory is proposed as a better explanatory paradigm of asset prices behavior. We merely wish to emphasize the need for revising the EMH and provide empirical evidence to this end.

The existing literature proposes several approaches for verifying whether a foreign exchange market is more or less efficient. In the remainder of this section we briefly review some of most significant findings.

From an econometric standpoint, Cornell (1977), Frankel (1980), Chiang and Jiang (1995), and Zhou (1996) examine whether the current spot, the forward rate or the futures price can be used as an unbiased predictor of the spot rate itself at some future date. From the same point of view, it is possible to use the recent time series tools of cointegration, ARCH and GARCH techniques to detect possible market inefficiencies. Kao and Ma (1992), Leachman and El Shazly (1992), Chan, Gup and Pan (1992) and Alexakis and Apergis (1996) utilize such methodologies.

A more operative approach consists of devising certain trading rules concerning these markets and determining their profitability, as in Taylor (1992), Levich and Thomas (1993), and Kho (1992).

A third class of techniques looks for deterministic nonlinear and chaotic dynamics in foreign currency market data. Hsieh (1988, 1992), and

Bleaney and Mizen (1996) follow these methodologies.

Finally, a recent "inter disciplinary" approach is the fractal one which is linked to both stochastic and deterministic aspects of the underlying process generating the price changes. The tools of fractal analysis are employed by Liu and Hsueh (1993), Fang, Lai and Lai (1994), Evertsz (1995a, 1995b), Evertsz and Berkner (1995), Van de Gucht, Dekimpe and Kwok (1996), Corazza, Malliaris and Nardelli (1997) and in this article. A detailed presentation of these techniques is given in Shubik (1997).

III. Theoretical Aspects

The current literature proposes different stochastic processes to describe the behavior of financial returns. The most common approaches are the fractional Brownian motion (fBm), and some of the Pareto-Levy stable (PLs) distribution sub-families. In general, these stochastic processes can be characterized by the same Hurst exponent, $H \neq 0.5$, as explained in Taqqu (1986), Evertsz (1995a, 1995b), and Evertsz and Berkner (1995). In fact, if such stochastic processes are independently and identically distributed with exponentially decaying power-law tails, as for example the PLs, then $H \in (0.5, 1)$, whereas if they are identically, but not independently distributed, as for example the fBm, then $H \in (0, 1)$.[6]

In order to conduct our analysis and consequently to test the MultiFractal Market Hypothesis (MFMH), we need a set of mathematical and statistical tools to formally define and estimate the long-term dependence of asset returns and to determine the value of the Hurst exponent. In particular, in this section we first define the fBm and PLs motions and present some of their properties. Second, we describe some tests for detecting long-term memory in time series and we introduce some algorithms for estimating the Hurst exponent, H.

A. Fractional and MultiFractional Brownian Motion

The fBm is a term coined by Mandelbrot and Van Ness (1968) to

6. Notice that the interval (0.5, 1) is obtained as the intersection of the ones characterizing each of the different PLs distribution sub-families. Taqqu (1986) includes in these subfamilies, the symmetric, the fractional and the log-fractional one, among others.

describe an almost everywhere continuous Gaussian stochastic process of index $H \in (0,1)$, $\{B_H(t),\ t \geq 0\}$, defined by a Riemann-Liouville stochastic integral, such that $B_H(0) = 0$ with probability 1, and that $B_H(t_2) - B_H(t_1) \sim N(0,\ \sigma^{2H}(t_2 - t_1)^{2H})$, with $0 \leq t_1 < t_2 < +\infty$ and $\sigma > 0$. In particular, if $H \neq 0.5$ then the increments are stationary but not independent, and they show a long-term memory depending on both H and $t_2 - t_1$. If $H \in (0, 0.5)$, there is a negative dependence between the increments. In this case the stochastic process has an anti-persistent behavior. If $H \in (0.5, 1)$, there is a positive dependence between the increments and in this case the process has a persistent behavior. The case $H = 0.5$ is the sBm that has independent increments. Moreover, this stochastic process is statistically self-similar, that is $\{B_H(t),\ t \geq 0\}$ and $\{a^{-H}B_H(at),\ t \geq 0\}$, with $a > 0$, have the same distribution law. Further details for the fBm can be found in Falconer (1990), Evertsz (1995a, 1995b), Evertsz and Berkner (1995) and Corazza, Malliaris and Nardelli (1997).

In 1995, Peltier and Levy (1995) proposed an extension of the fBm by substituting the constant over time Hurst exponent, H, with a suitable time dependent function, $H(t)$. Unlike the fBm, this new stochastic process, called multifBm (mfBm), allows us to formally model the irregularities of the process trajectory. As such, this stochastic process can be fruitfully utilized to describe non-stationarity in financial asset price variations.[7]

B. Pareto-Lévy Stable Stochastic Process

The PLs motion, originally introduced by Lévy (1925) as a generalization of the sBm, is a stochastic process, $\{L_\alpha(t),\ t \geq 0\}$, characterized by a distribution, $S_{\alpha,\beta}(\mu,\sigma)$, depending on four parameters: the so-called characteristic exponent $\alpha \in (0, 2]$, the skewness parameter $\beta \in [-1, 1]$, the location parameter $\mu \in (-\infty, +\infty)$, and the scale coefficient $\sigma \in [0, +\infty)$.[8] This stochastic process is such that $L_\alpha(0) = 0$ almost-surely, and its increments $L_\alpha(t_2) - L_\alpha(t_1)$, with $0 \leq t_1 < t_2 < +\infty$, whose distribution is $S_{\alpha,\beta}(0, (t_2 - t_1)^{1/\alpha})$, are independent and stationary. In particular, if α

7. For details, see Cheung and Lai (1993), Corazza (1996) and Belkacem, Levy and Walter (1996).

8. If $\alpha \in (0,1)$ the distribution does not have a finite mean or a finite variance. If $\alpha \in [1,2)$ the distribution has only a finite mean and if $\alpha = 2$, the distribution has both finite mean and finite variance.

$\in (0, 2)$ then the tails of such a process decay slower than the tails of an fBm process, and if $\alpha = 2$ it is possible to prove that $\{2^{-1/2} L_2(t), t \geq 0\} \equiv \{B_{0.5}(t), t \geq 0\}$, which is the sBm. Moreover, if the distribution $S_{\alpha,\beta}(\mu, \sigma)$ is symmetric, that is if $\beta = 0$, then the corresponding PLs process is statistically self-similar.[9] Taking $\{L_\alpha(t), t \geq 0\}$ and $\{a^{-1/\alpha} L_\alpha(at), t \geq 0\}$, with $a > 0$, results in the same distribution law. In such a case it is possible to prove that the Hurst exponent equals $H = 1/\alpha$.[10]

IV. Empirical Aspects

Although a large empirical literature exists confirming the presence of long-run memory or long-range dependence in asset prices, there are no universally accepted quantitative methodologies that make it possible to detect such long-term dependence in (finite) time series as argued by Taqqu, Teverovsky and Willinger (1995). Moreover, some of the methodologies used show considerable limitations. Thus, in order to overcome the shortcomings of each methodology, we follow two different inferential approaches and compare the corresponding results. The methodologies employed are the classical modified range over standard deviation statistic, *R/S,* and the periodogram approach.[11]

A. Tests for Long-Term Dependence

i. The Modified R/S Test

Lo (1991) proposes a modification of a test based on the classical range over standard deviation statistic, *R*/*S*. To test for no long-term dependence in financial time series consider:

$$Q_r(q) = \frac{R_T / S_T(q)}{\sqrt{T}}, \tag{4.1}$$

9. From a financial standpoint it is not restrictive to assume that $\beta = 0$. In fact, most of the skewness parameters estimated from asset returns time series, though different from 0, are quite close to it.

10. See Taqqu (1986) and Corazza, Malliaris and Nardelli (1997).

11. We are grateful to an anonymous referee for suggesting that we use both methodologies.

where T is the time series size, q is the possible short-term dependence (integer) length, $R_T(q)$ is the sample range of partial sums of deviations of the time series from its sample mean, and $S_T(q)$ is the modified standard deviation of the time series including the autocovariances weighted up to lag q. This new methodology is described in detail in both Lo (1991) and Campbell, Lo and MacKinlay (1997). Precisely, this statistic is able to test the null hypothesis of no long-term dependence.[12] In particular, unlike the corresponding statistic based on the classical *R*/*S*, it is robust to short-term memory, conditional heteroscedasticity, and non-normal innovation. Furthermore, it also has well-defined distributional properties as described in Lo (1991) and Campbell, Lo and MacKinlay (1997), although the related (asymptotic) distribution is neither standard, nor easily tractable.

Of course, this statistic is crucially influenced by the statistical structure of short-term dependence. In order to accommodate this aspect, we apply two different approaches. In the first approach we specify in a nonparametric way the short-term memory structure determining the optimal value of q by the use of the Andrews' (1991) data-dependent rule $q^* = [(3T/2)^{1/3}[\rho/(1-\rho^2)]^{2/3}]$, where the operator $[\cdot]$ denotes the greatest integer less than or equal to the argument, and ρ is the sample first-order autocorrelation coefficient. In the second approach, we take into account the remarks of Lo (1991) and Jacobsen (1996) stating that, in general, there is little guidance in determining the optimal value of q. In this article, we follow the Jacobsen's (1996) procedure, and perform the test in two steps. First, we impose some specific models for the short-term dependence structure, namely an AR(1) one and a MA(1) one.[13] Second, we apply the statistic $Q_T(q^*)$, with $q^* = 0$, to the time series of the corresponding residuals.

Finally, by using the fractiles of the distribution of $Q_T(q)$ as in Lo

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12. Notice that a rejection of such a null hypothesis does not necessarily imply that long-range dependence is present but, merely, that the underlying stochastic process does not simultaneously satisfy all the conditions stated by Lo (1991). However, such conditions are satisfied by many of the recently proposed stochastic processes for long-term dependence.

13. Notice that such an arbitrary way of choosing an AR(1) model and a MA(1) one is not particularly restrictive because, in general, such models are standard for handling short-term memory in financial returns time series.

(1991), it is possible to determine critical values for different significance levels in this two-sided test. At 10, 5 and 1 percent they are 1.747, 1.862, and 2.098, respectively.

ii. The Periodogram-based Test

Lobato and Savin (1998) employ a suitable approximation to the Lagrange multiplier test in order to develop the no long-term dependence in the following time series statistic that is based on a periodogram and is given as such:

$$LM_T(m) = m\left(\frac{\sum_{j=1}^{m} v_j I(\lambda_j)}{\sum_{j=1}^{m} I(\lambda_j)}\right), \tag{4.2}$$

where m is an (integer) bandwidth,

$$v_j = \ln(j) - \left[\sum_{j=1}^{m} \ln(j)\right] \Big/ m,$$

and

$$I(\lambda_j) = \left[\left|\sum_{t=1}^{T} x_t \exp(it\lambda_j)\right|\right] \Big/ (2\pi T),$$

is the periodogram computed at frequency $\lambda_j = (2\pi j)/T$, in which x_t, with $t = 1,..., T$, is the time series, and $i = \sqrt{-1}$. More specifically, this statistic tests the null hypothesis H_0: $H = 0.5$ rather than the alternative one H_A: $H \neq 0.5$. Moreover, this test is characterized by a well-known and quite tractable (asymptotic) distribution which is the χ_1^2.

Of course, in this statistic, the bandwidth m plays a crucial role. In order to determine its optimal value, we need to verify that certain proper assumptions hold (such as the Gaussianity of x_t, with $t = 1, ..., T$). We use the iterative algorithm presented in Delgado and Robinson (1996) to estimate m. We could also apply the widely used “rule of thumb" that sets $m = \sqrt{T}$.

Finally, by using the fractiles of the χ^2_1 distribution, it is possible to determine the critical values for different significance levels for this two-sided test.

B. Procedure for Estimating the Hurst Exponent

i. The Modified R/S Estimation Procedure

The Hurst exponent is linked to the modified *R/S* statistic by $\lim_{t\to+\infty} E[R_T/S_T(q)]/(aT^H) = 1$, with $a > 0$. With this link it is possible to obtain the following approximate relationship: $\ln\{E[R_T/S_T(q)]\} \cong \ln(a) + H\ln(t)$. In order to estimate the value of the Hurst exponent, H, we have modified and improved the standard techniques described in Peters (1991, 1994), Corazza (1996) and Corazza, Malliaris and Nardelli (1997).

To do so, we first determine a series of estimates of the Hurst exponent $\{H_j,\ j = 1, ..., T^* < T\}$ by fitting an ordinary least square regression between $\{\ln[R_{T,l}/S_{T,l}(q)]\},\ l = 1, ..., j\}$ and $\{\ln(l),\ l = 1, ..., j\}$, for every $j = 2, ..., T^*$, where $R_{T,l}$ and $S_{T,l}(q)$ are quantities related to R_T and $S_T(q)$ respectively. Then, we choose the optimal estimate in this series. Figures 1A and 1B illustrate the corresponding results for some of the analyzed time series by plotting H_j versus j, with $j = 2,..., T^*$. In particular, this estimation procedure is robust, although possibly subject to bias, when the data generating process follows a highly non-normal distribution as argued by Lo (1991), Cheung and Lai (1993), Robinson (1994b), and Campbell, Lo and MacKinlay (1997). It is possible to prove its almost-sure convergence for stochastic processes with infinite variance. Consider for example the PLs distribution with $\alpha \in (0,2)$. Furthermore, Robinson (1994b) argues that the *R/S* estimation procedure is suboptimal when the data generating process follows a Gaussian distribution because such a procedure does not depend on second moments.

Overall, the *R/S*-based estimation procedure described in this section offers the possibility to estimate the Hurst exponent without complete information, and without strong a priori assumptions on the distributional properties of the considered stochastic process.[14]

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14. See Pancham (1994), Peltier and Levy (1994) and Taqqu, Teverovsky and Willinger (1995) for a different methodology.

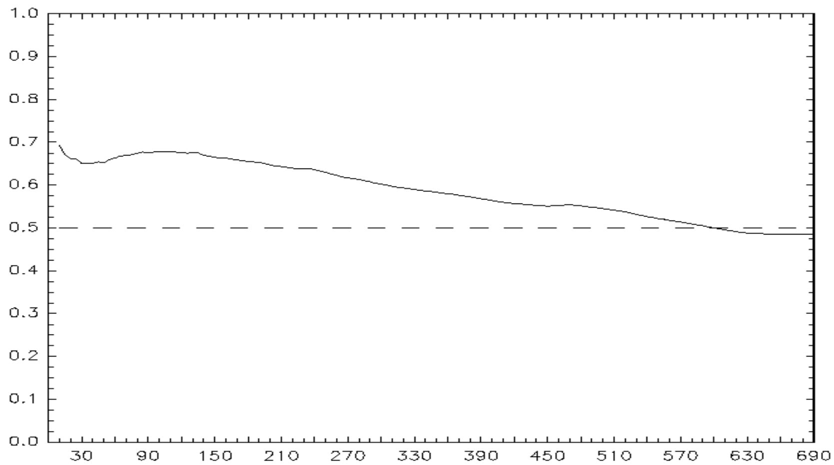

FIGURE 1A—*H* versus *j* for Canadian Dollar S. (08/76-01/82): The AR(1)case (T^*=690).

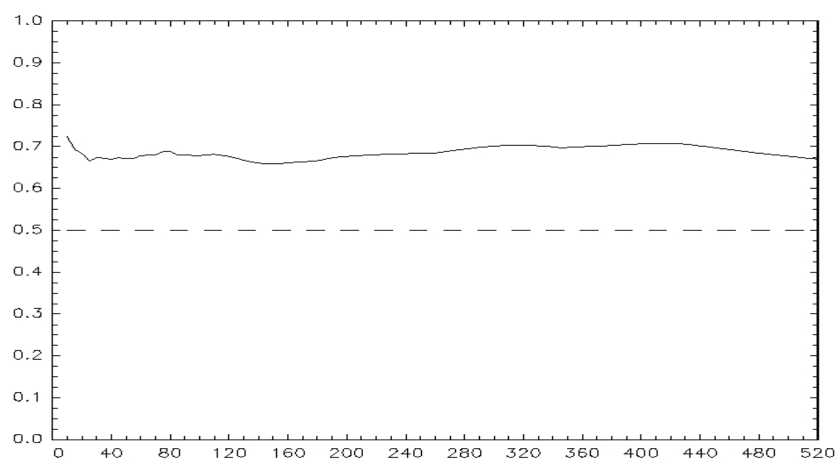

FIGURE 1B—*H* versus *j* for British Pound F. (06/72-07/76): The q = # case (T^*=520)

ii. The Periodogram-based Estimation Procedure

From a spectral density point of view, the Hurst exponent is linked to the discretely averaged periodogram:

$$F(\lambda) = 2\pi \left[\sum_{j=1}^{\lfloor \lambda T/2\pi \rfloor} I(\lambda_j) \right] \Big/ T.$$

Starting from this relationship, Robinson (1994a) proposed the following closed form semi-parametric estimator for *H*:

$$H(m,r) = 1 - \frac{1}{2\ln(r)} \ln\left[\frac{F(r\lambda_m)}{F(\lambda_m)}\right], \tag{4.3}$$

where *m* is the bandwidth introduced earlier and $r \in (0,1)$ is a suitable user-chosen variable. In particular, under the hypothesis that the data-generating process follows a Gaussian distribution, it is possible to prove that this estimator is consistent and that it has well-defined (asymptotic) distributional properties both normal and non-normal, depending on the estimated value of $H(m, r)$.[15, 16]

Of course *r*, plays a crucial role in this estimator. In particular, if some proper assumptions hold, among them the restrictions that $H \in (0.5, 0.75)$, then it is possible to determine its optimal value as discussed in Lobato and Robinson (1996). Thus, since both *m* and *r* depend on $H(m, r)$, in order to optimally estimate the Hurst exponent, we must determine a suitable series of converging estimates of *H*, $\{H_j(m_j, r_j), j = 1, ..., J\}$. This can be done using the iterative algorithm proposed in Delgado and Robinson (1996). Figure 1C illustrates the corresponding results for one of the analyzed time series by plotting $H_j(m_j, r_j)$ versus *j*, with $j = 1, ..., J$.

V. Data Set and Descriptive Statistics

The data we analyze are the time series of the daily returns using closing prices of exchange rates expressed in US Dollars, that is, $100\{\ln[P(t+1)] - \ln[P(t)]\}$. We use data from June 1972 to September 1994, for the following five spot and (nearby) futures foreign currency markets:

15. See Robinson (1994a).

16. See Lobato and Robinson (1996).

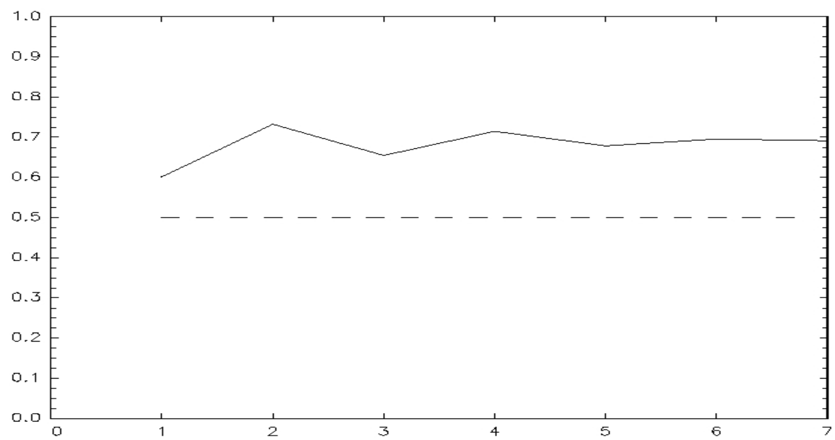

FIGURE 1C—*H versus j* for German Mark F. (06/72-07/76): The Periodogram case (J=7)

British Pound, Canadian Dollar, German Mark, Swiss Franc and Japanese Yen. In particular, in order to implement our multifractal analysis, we assume that the dynamic Hurst exponent $H(t)$is a stepwise constant function whose intervals are determined by splitting up each time series into four non-overlapping sub-periods: June 1972 to July 1976; August 1976 to January 1982; February 1982 to June 1987; and July 1987 to September 1994. The choice of these four time sub-periods is driven by the (relative) homogeneity of the economic and political conditions in each geographical region.

In table 1A to table 1F, we report some standard descriptive statistics. The quantities reported indicate the number of observations, the minimum and maximum values of the time series, the means, the medians, the standard deviations, the skewness, and the kurtosis.

Generally, all the considered time series qualitatively denote to some degree a departure from normality. This is evidenced by the medians that differ from the corresponding means, skewness values, and particularly, kurtosis values. These departures are also confirmed by the performance of a simple χ^2-type test for distribution fitting, which rejects the null hypothesis of normality for all the time series at 1% significance level.

From a short and medium-term autocorrelation point of view, we investigate the sample autocorrelation function up to lag 22 (about a one-month trading period). In general, with the exception of certain time

TABLE 1. Descriptive Statistics

Time Period	N. Obs.	Min	Max	Mean	Median	St. Dev.	Skew.	Kurtosis
A. British Pound								
i. Spot								
06/72–07/76	1033	–3.0589	3.1812	–0.0369	0.0000	0.4470	–0.2436	9.0562
08/76–01/82	1382	–4.6623	3.7496	0.0037	0.0057	0.6302	–0.6097	6.8748
02/82–06/87	1377	–3.0175	4.5942	–0.0110	0.0000	0.7392	0.4208	3.3990
07/87–09/94	1894	–4.0900	3.2656	–0.0010	0.0000	0.7313	–0.2248	2.5073
06/72–09/94	5686	–4.6623	4.5942	–0.0088	0.0000	0.6659	–0.0857	4.3772
ii. (*Nearby*) *Futures*								
06/72–07/76	1045	–2.2103	2.8738	–0.0374	0.0000	0.4694	–0.5182	5.1612
08/76–01/82	1384	–3.4467	3.6057	0.0044	0.0000	0.6541	–0.4975	4.0314
02/82–06/87	1369	–2.7369	4.5529	–0.0113	0.0000	0.7714	0.4842	3.4082
07/87–09/94	1844	–4.4760	3.4748	–0.0010	0.0215	0.7714	–0.2628	2.5643
06/72–09/94	5642	–4.4760	4.5529	–0.0089	0.0000	0.6963	–0.0749	3.7279

TABLE 1. Descriptive Statistics

B. Canadian Dollar								
i. Spot								
06/72–07/76	1034	–1.5467	1.1957	0.0006	0.0000	0.1467	–0.5417	17.3788
08/76–01/82	1382	–1.8677	0.8678	–0.0148	–0.0212	0.2439	–0.4324	3.9179
02/82–06/87	1374	–1.6555	1.4323	–0.0078	–0.0122	0.2571	–0.2355	5.0739
07/87–09/94	1895	–1.9088	1.9971	–0.0006	0.0119	0.2735	–0.3115	4.3041
06/72–09/94	5685	–1.9088	1.9971	–0.0056	0.0000	0.2436	–0.3453	5.4804
ii. (*Nearby*) *Futures*								
06/72–07/76	1045	–1.1974	0.7754	0.0006	0.0000	0.1622	–0.2832	6.6126
08/76–01/82	1384	–1.1939	1.1851	–0.0144	–0.0118	0.2643	0.0482	1.5593
02/82–06/87	1369	–1.7946	1.6525	–0.0079	–0.0122	0.2745	–0.1552	5.1085
07/87–09/94	1844	–1.7811	1.9916	–0.0005	0.0230	0.3026	–0.5787	4.1179
06/72–09/94	5642	–1.7946	1.9916	–0.0055	0.0000	0.2651	–0.3262	4.5511

TABLE 1. (Continued)

C. German Mark								
i. Spot								
06/72–07/76	1033	–4.3193	6.0458	0.0219	0.0000	0.6695	0.5869	12.4245
08/76–01/82	1381	–7.0967	3.1639	0.0060	0.0000	0.6367	–0.7251	13.3480
02/82–06/87	1374	–3.2019	4.9899	0.0177	0.0000	0.7338	0.4353	2.5384
07/87–09/94	1894	–3.4661	3.1659	0.0090	0.0000	0.7149	–0.0533	1.8184
06/72–09/94	5682	–7.0967	6.0458	0.0127	0.0000	0.6933	0.0658	5.7780
ii. (*Nearby*) *Futures*								
06/72–07/76	1046	–1.8976	3.8037	0.0219	0.0000	0.5649	0.7705	4.1370
08/76–01/82	1384	–3.6945	3.4361	0.0064	0.0000	0.6416	0.2582	3.3015
02/82–06/87	1369	–3.2351	4.8321	0.0177	0.0000	0.7647	0.5001	2.5882
07/87–09/94	1844	–3.3125	3.6013	0.0088	0.0000	0.7392	–0.0997	1.7442
06/72–09/94	5643	–3.6945	4.8321	0.0128	0.0000	0.6932	0.2497	2.7370

TABLE 1. (Continued)

D. Japaneese Yen								
i. Spot								
06/72–07/76	1033	–6.2566	8.7260	0.0035	0.0000	0.4816	3.9312	133.7888
08/76–01/82	1382	–5.2644	3.5703	0.0182	–0.0224	0.6890	0.1337	4.3757
02/82–06/87	1374	–2.3846	5.4055	0.0321	0.0000	0.6558	0.7768	5.2895
07/87–09/94	1894	–4.0991	3.8777	0.0204	–0.0220	0.6953	0.0755	3.5482
06/72–09/94	5683	–6.2566	8.7260	0.0196	0.0000	0.6502	0.5588	11.6583
ii. (*Nearby*) *Futures*								
06/72–07/76	1046	–5.6660	5.5346	0.0027	0.0000	0.5526	–0.2844	26.8716
08/76–01/82	1384	–2.7504	4.8110	0.0189	0.0000	0.7290	0.5778	2.8046
02/82–06/87	1369	–2.3653	5.3327	0.0320	0.0000	0.6677	0.7597	4.6245
07/87–09/94	1844	–4.2073	4.7533	0.0208	0.0000	0.7008	0.1364	3.8295
06/72–09/94	5643	–5.6660	5.5346	0.0197	0.0000	0.6751	0.3821	5.8022

TABLE 1. (Continued)

E. Swiss Franc								
i. Spot								
06/72–07/76	1033	–4.3367	3.7346	0.0427	0.0249	0.7248	0.1778	6.3538
08/76–01/82	1379	–7.0054	4.4466	0.0210	0.0000	0.8356	0.4140	8.8837
02/82–06/87	1372	–3.9302	5.3094	0.0145	0.0000	0.8187	0.3038	2.6768
07/87–09/94	1891	–3.5750	3.4613	0.0087	0.0000	0.7797	0.0465	1.4185
06/72–09/94	5675	–7.0054	5.3094	0.0193	0.0000	0.7938	0.0003	4.6543
ii. (Nearby) Futures								
06/72–07/76	1047	–3.2377	4.6886	0.0424	0.0000	0.6461	0.4319	5.0734
08/76–01/82	1384	–3.9371	4.3620	0.0213	–0.0173	0.8098	0.4728	2.8961
02/82–06/87	1369	–3.6919	5.5361	0.0144	0.0000	0.8640	0.4471	2.5081
07/87–09/94	1844	–3.6227	3.1341	0.0086	0.0000	0.8074	–0.0210	1.2281
06/72–09/94	5644	–3.9371	5.5361	0.0194	0.0000	0.7953	0.2906	2.5401

TABLE 1. (Continued)

	06/72–07/76	08/76–01/82	02/82–06/87	07/87–09/94	06/72–09/94
F. Short Term Dependence Analysis					
British Pound (Spot)	9, 14	9	11	6, 10, 18	9, 11, 20
British Pound (Fut.)	9	–	6	6, 15, 18	1, 15
Canadian Dollar (Spot)	1, 2, 7, 10	1, 5	1, 2, 3, 12, 16	4, 16	1, 4, 5, 7, 16
Canadian Dollar (Fut.)	–	5	1, 2, 3, 12, 13	12, 15	1, 2
German Mark (Spot)	2, 3, 5, 7, 9, 10, 11, 14	3, 10	10	10	3, 9, 10
German Mark (Fut.)	10, 13, 18	10	3, 8, 11, 16	15	15, 20
Japanese Yen (Spot)	1, 10, 20, 21	1, 9, 10, 13	3, 5, 6	6, 10, 14	9, 10
Japanese Yen (Fut.)	4	10, 20, 21	6	6, 10, 14	8, 9, 10, 14, 21
Swiss Franc (Spot)	2, 7	10	–	10	9, 12
Swiss Franc (Fut.)	1, 8	1, 2, 20	16	2, 15	15

series, such an autocorrelation structure is negligible. In table 1F, we report the lag(s) for which the corresponding autocorrelation coefficient is significantly different from 0 at the 5% significance level for each time series under observation.[17]

Finally, some authors, such as Lobato and Savin (1998), suggest that evidence of long-term memory could be spuriously caused by non-stationarity in the time series itself. To test for non-stationarity, we perform the basic Dickey-Fuller test and its properly augmented version.[18] For all the considered series, both tests reject the null hypothesis of non-stationarity (more precisely the tests reject the presence of a unit root in the autoregressive representation) at the 2% significance level.[19]

VI. Empirical Results of MultiFractal Analysis[20]

The empirical results obtained are reported in table 2A to table 2E. In particular, the results relative to each of the considered single time periods are presented in four rows. The first three rows are devoted to the modified *R/S*-based approach, and the fourth row is devoted to the periodogram-based approach. In the columns labeled "*" we report the information concerning the assumed short-term dependence structure (in the first three rows relative to each period), and the bandwidth value (in the fourth row relative to each period). In the columns labeled "H_0" we report the results of the test for no long-term dependence (acceptance or non-rejection is indicated by "A", rejection is indicated by its significance level), and in the columns labeled "*H*" we report the values of the Hurst exponent.[21]

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17. Such as the the Canadian Dollar spot, the Canadian Dollar futures, the German Mark spot, the Japanese Yen spot, and the Swiss Franc futures in some sub and full-sample periods.

18. For more details see Dickey and Fuller (1979, 1981).

19. The 2% significance level is the lowest boundary of the significance levels tabulated in Dickey and Fuller (1979).

20. Statistical computations were performed by Marco Corazza.

21. Notice that, although for completeness of exposition we also report the cases when the null hypothesis is rejected at the 20% significance level, practically we consider such rejections as acceptances in table 2F.

TABLE 2. Multifractal Analysis

Spot				(*Nearby*) *Futures*			
Time Period	*	H_0	H	Time Period	*	H_0	H
A. British Pound							
06/72–07/76	$q = 1$	5%	0.5991	06/72–07/76	$q = 1$	5%	0.7079
	AR(1)	10%	0.6798		AR(1)	5%	0.7057
	MA(1)	10%	0.6838		MA(1)	5%	0.7474
	$m = 258$	A	0.5828		$m = 261$	A	0.5816
08/76–01/82	$q = 1$	1%	0.5466	08/76–01/82	$q = 2$	5%	0.5499
	AR(1)	20%	0.6408		AR(1)	20%	0.6253
	MA(1)	20%	0.6343		MA(1)	20%	0.6027
	$m = 326$	A	0.5139		$m = 326$	5%	0.4844
02/82–06/87	$q = 1$	5%	0.6446	02/82–06/87	$q = 0$	5%	0.6424
	AR(1)	A	0.5990		AR(1)	A	0.5906
	MA(1)	A	0.5881		MA(1)	A	0.5940
	$m = 324$	A	0.4821		$m = 323$	A	0.4565
07/87–09/94	$q = 2$	20%	0.5022	07/87–09/94	$q = 2$	10%	0.4854
	AR(1)	A	0.6255		AR(1)	A	0.6181
	MA(1)	A	0.6046		MA(1)	A	0.5942
	$m = 419$	A	0.5206		$m = 410$	A	0.5137
06/72–09/94	Q=2	1%	0.6780	06/72–09/94	$q = 2$	1%	0.6778
	AR(1)	1%	0.6336		AR(1)	5%	0.6658
	MA(1)	1%	0.6499		MA(1)	1%	0.6228
	$m = 1009$	5%	0.5399		$m = 1003$	A	0.5305

TABLE 2. (Continued)

B. Canadian Dollar

06/72–07/76	$q = 4$	10%	0.6175	06/72–07/76	$q = 1$	20%	0.6273
	AR(1)	A	0.6523		AR(1)	20%	0.6089
	MA(1)	10%	0.6083		MA(1)	20%	0.6026
	m=258	20%	0.6216		$m = 261$	A	0.5287
08/76–01/82	$q = 4$	20%	0.4449	08/76–01/82	$q = 2$	10%	0.4497
	AR(1)	20%	0.4890		AR(1)	10%	0.4671
	MA(1)	20%	0.4384		MA(1)	10%	0.4476
	$m = 326$	10%	0.6216		$m = 326$	20%	0.5855
02/82–06/87	$q = 3$	10%	0.6123	02/82–06/87	$q = 5$	20%	0.5947
	AR(1)	A	0.6116		AR(1)	A	0.5916
	MA(1)	20%	0.5805		MA(1)	A	0.5601
	$m = 324$	A	0.5244		$m = 323$	20%	0.4430
07/87–09/94	$q = 1$	10%	0.5873	07/87–09/94	$q = 2$	10%	0.5072
	AR(1)	A	0.6046		AR(1)	20%	0.5178
	MA(1)	A	0.5833		MA(1)	A	0.5130
	$m = 419$	A	0.4929		$m = 410$	A	0.4217
06/72–09/94	$q = 4$	5%	0.5937	06/72–09/94	$q = 4$	10%	0.5896
	AR(1)	10%	0.6080		AR(1)	20%	0.5997
	MA(1)	10%	0.5957		MA(1)	20%	0.5935
	$m = 1009$	A	0.5494		$m = 1003$	A	0.5033

TABLE 2. (Continued)

C. German Mark							
06/72–07/76	$q = 0$	20%	0.7105	06/72–07/76	$q = 2$	5%	0.6999
	AR(1)	20%	0.7141		AR(1)	10%	0.7440
	MA(1)	20%	0.7105		MA(1)	5%	0.6919
	$m = 258$	A	0.6199		$m = 261$	1%	0.6910
08/76–01/82	$q = 1$	5%	0.6123	08/76–01/82	$q = 2$	5%	0.5918
	AR(1)	20%	0.5825		AR(1)	10%	0.5620
	MA(1)	20%	0.5738		MA(1)	10%	0.5345
	$m = 326$	A	0.5810		$m = 326$	20%	0.5729
02/82–06/87	$q = 0$	1%	0.6519	02/82–06/87	$q = 2$	1%	0.6208
	AR(1)	20%	0.5711		AR(1)	A	0.6391
	MA(1)	A	0.4088		MA(1)	A	0.6036
	$m = 324$	10%	0.5799		$m = 323$	10%	0.5797
07/87–09/94	$q = 0$	A	0.6356	07/87–09/94	$q = 2$	A	0.6157
	AR(1)	A	0.6342		AR(1)	A	0.6340
	MA(1)	A	0.6356		MA(1)	A	0.6140
	$m = 419$	A	0.4930		$m = 410$	A	0.4753
06/72–09/94	$q = 0$	1%	0.6600	06/72–09/94	$q = 2$	1%	0.6563
	AR(1)	10%	0.6175		AR(1)	10%	0.6161
	MA(1)	10%	0.6179		MA(1)	10%	0.6120
	$m = 1009$	5%	0.5832		$m = 1003$	5%	0.5795

TABLE 2. (Continued)

D. Japanese Yen							
06/72–07/76	q = 2	A	0.6087	06/72–07/76	q = 2	A	0.6133
	AR(1)	A	0.6546		AR(1)	A	0.6317
	MA(1)	A	0.6022		MA(1)	A	0.6103
	m = 258	20%	0.6133		m = 261	A	0.5800
08/76–01/82	q = 2	1%	0.7173	08/76–01/82	q = 1	1%	0.7321
	AR(1)	10%	0.6572		AR(1)	10%	0.6418
	MA(1)	10%	0.6340		MA(1)	10%	0.6389
	m = 326	A	0.5724		m = 326	A	0.5509
02/82–06/87	q = 1	5%	0.6314	02/82–06/87	q = 1	5%	0.6330
	AR(1)	10%	0.6589		AR(1)	10%	0.6581
	MA(1)	10%	0.6616		MA(1)	10%	0.6129
	m = 324	5%	0.6512		m = 323	5%	0.6145
07/87–09/94	q = 1	A	0.6211	07/87–09/94	q = 1	A	0.6200
	AR(1)	A	0.6287		AR(1)	A	0.6267
	MA(1)	20%	0.6215		MA(1)	A	0.6194
	m = 419	A	0.5026		m = 410	A	0.4897
06/72–09/94	q = 1	1%	0.6245	06/72–09/94	q = 1	5%	0.6200
	AR(1)	1%	0.6297		AR(1)	5%	0.6224
	MA(1)	1%	0.5320		MA(1)	5%	0.6199
	m = 1009	1%	0.6077		m = 1003	10%	0.5849

TABLE 2. (Continued)

E. Swiss Frank

06/72–07/76	$q = 0$	20%	0.6877	06/72–07/76	$q = 3$	10%	0.6670
	AR(1)	20%	0.6925		AR(1)	10%	0.7114
	MA(1)	20%	0.6877		MA(1)	10%	0.6593
	$m = 258$	20%	0.5806		$m = 261$	20%	0.6127
08/76–01/82	$q = 2$	5%	0.6224	08/76–01/82	$q = 3$	10%	0.6119
	AR(1)	20%	0.6451		AR(1)	20%	0.6386
	MA(1)	20%	0.6268		MA(1)	A	0.5897
	$m = 325$	A	0.5466		$m = 326$	5%	0.5517
02/82–06/87	$q = 1$	5%	0.6604	02/82–06/87	$q = 2$	5%	0.6436
	AR(1)	20%	0.5512		AR(1)	20%	0.5417
	MA(1)	20%	0.5470		MA(1)	20%	0.5131
	$m = 325$	A	0.5405		$m = 323$	A	0.5246
07/87–09/94	$q = 1$	20%	0.6306	07/87–09/94	$q = 2$	20%	0.6225
	AR(1)	20%	0.6362		AR(1)	A	0.6421
	MA(1)	20%	0.6303		MA(1)	20%	0.6217
	$m = 418$	A	0.4988		$m = 410$	A	0.4819
06/72–09/94	$q = 1$	1%	0.6550	06/72–09/94	$q = 2$	1%	0.6517
	AR(1)	10%	0.5991		AR(1)	10%	0.6015
	MA(1)	10%	0.5956		MA(1)	10%	0.5933
	$m = 1008$	A	0.5473		$m = 1003$	A	0.5495

TABLE 2. (Continued)

Time Period	B. Pound		C. Dollar		G. Mark		J. Yen		S. Franc	
	Spot	Fut.	Spot	Fut.	Spot	Fut.	Spot	Fut.	Spot	Fut.
F. Comparative MultiFractal Analysis										
6/72-07/76	fBm	fBm	fBm	Pls	PLs	fBm	PLs	PLs	PLs	fBm
	fBm	fBm	PLs	PLs	PLs	fBm	PLs	PLs	PLs	fBm
	fBm	fBm	fBm	PLs	PLs	fBm	PLs	PLs	PLs	fBm
	PLs	PLs	PLs	PLs	PLs	fBm	PLs	PLs	PLs	PLs
08/76-01/82	fBm	fBm	?	fBm	fBm	fBm	fBm	fBm	fBm	fBm
	PLs	PLs	?	fBm	PLs	fBm	fBm	fBm	PLs	PLs
	PLs	PLs	?	fBm	PLs	fBm	fBm	fBm	PLs	PLs
	PLs	fBm	fBm	PLs	PLs	PLs	PLs	PLs	PLs	fBm
02/82-06/87	fBm	fBm	fBm	PLs	fBm	fBm	fBm	fBm	fBm	fBm
	PLs	PLs	PLs	PLs	PLs	PLs	fBm	fBm	PLs	PLs
	PLs	PLs	PLs	PLs	?	PLs	fBm	fBm	PLs	Pls
	?	?	PLs	?	fBm	fBm	fBm	fBm	PLs	Pls
07/87-09/94	PLs	fBm	fBm	fBm	PLs	PLs	PLs	PLs	PLs	PLs
	PLs	PLs	PLs	PLs	PLs	PLs	PLs	PLs	PLs	PLs
	PLs	PLs	PLs	PLs	PLs	PLs	PLs	PLs	PLs	PLs
	PLs	PLs	?	?	?	?	PLs	?	?	?
06/72-09/94	fBm	fBm	fBm	fBm	fBm	fBm	fBm	fBm	fBm	fBm
	fBm	fBm	fBm	PLs	fBm	fBm	fBm	fBm	fBm	fBm
	fBm	fBm	fBm	PLs	fBm	fBm	fBm	fBm	fBm	fBm
	fBm	PLs	PLs	PLs	fBm	fBm	fBm	fBm	PLs	PLs

Generally, from the results reported in Table 2A to Table 2E, we observe that for the 66% of the considered time periods, both the modified *R/S*-based and the periodogram-based-tests qualitatively agree to accept or reject the null hypothesis of no long-term memory.[22]

We also wish to note that, in general, the estimates of H based on the modified *R/S* approach are greater than the corresponding estimates based on the periodogram approach. This is in accordance with the findings of Mandelbrot and Wallis (1969) and Jacobsen (1996), which confirm that the modified *R/S*-based estimation procedure overestimates the value of H when the true value is lower than 0.72 (as it seems to be in the majority of our cases).

Again, for all time periods and for both spot and (nearby) futures foreign currency markets, the corresponding value of the dynamic Hurst exponent $H(t)$ is neither equal to 0.5 nor constant over time. This provides us with important empirical evidence for the MFMH or, at least, for the need to revise the EMH. In particular, the dynamic dimension is well supported by the test for no long-term dependence results. In fact, both the spot and (nearby) futures foreign currency markets are characterized over time by different underlying stochastic processes: the fBm, the PLs motion and an undetectable one.[23]

Almost all the fBms describing the stochastic behavior of a wide percentage of the time sub-periods show a persistent long-term dependence, that is $H \in (0.5, 1)$, and all the PLs motions describing the stochastic behavior of another wide percentage of the time sub-periods are distinguished by the non-finiteness of the variance, that is by $\alpha \in (1, 2)$ (by $\alpha = 1/H$). Coupling both these aspects, that is, long-term dependence/independence and variance finiteness/non-finiteness, it follows that the structure of financial risk can vary widely from one time sub-period to the next.

In general, the spot and the (nearby) futures foreign currency

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22. There are instances when at least two of the three sub-cases of the modified R/S-based approach (q = #, AR(1), and MA(1)) qualitatively agree with the only accept/reject decision given by the periodogram-based approach.

23. For these processes jointly characterized by $H \in (0,0.5)$ and long-term independence, some authors, such as Evertsz (1995a, 1995b), suggest suitable mixtures of fBms and PLs motions. Others, like Zou (1996) suggest that some proper PLs distribution sub-families, such as a fractional distribution may be suitable. These issues have not been settled and are beyond the scope of this work.

markets for each currency are characterized by similar dynamic stochastic structures, especially from a short and long-term dependence/independence point of view.

VII. Economic Interpretations

In general, all the analyzed foreign currency markets exhibit a behavior over time influenced by their Hurst exponent and by their long-term independence/dependence. This behavior provides empirical support for the MFMH as a reasonable extension of the EMH. In fact, different stochastic processes describe the foreign currency markets during various periods. We distinguish three phases characterizing the conjectured MFMH (instead of the two standard ones): a "regular" phase, a new phase that we identify as "semi-regular" and an "irregular" phase.

The "regular" phase is associated with the fBm via long-term dependence, that is, with the Hurst exponent $H \in (0.5, 1)$. In fact, the characteristics of the financial risk described by the corresponding distributional law are such as to permit a relatively simple matching between the demand and supply for two reasons:

First, the statistical self-similarity characterizing the fBms guarantees that the risk associated with investments of different horizon lengths t and at, with $a > 0$, are evaluated in the same proportion by their corresponding investors. Actually, $\{B_H(t), t \geq 0\}$ and $\{a^{-H}B_H(at), t \geq 0\}$, with $a > 0$, have the same distributional law.[24] Because of this, the demands and supplies of these investors with different horizon lengths match, and thus ensure a certain liquidity for the foreign currency markets. Notice that the statistical self-similarity implicitly asserts the existence of some relationships between the Hurst exponent, H, and the liquidity level.

Second, the long-term persistent memory distinguishing these foreign currency markets makes it possible to partially forecast future returns, and consequently, *ex ceteris paribus*, to manage a lower risk than in the classical independently and identically log-normally distributed

24. Notice that a^{-H} plays the role of a proportionality factor.

environment.[25] This lowering of long-run risk may explain the attractiveness of longer investment horizons by some investors in an fBm regime.

In particular, in order to explain such long-term persistent memory, we can conjecture that the analyzed foreign currency markets are characterized by the regular arrival of new information confirming underlying economic trends. Of course, this reduces the spread between the ability of economic agents to make optimal decisions and the complexity of decisions made under uncertainty.

The "semi-regular" phase is associated with the PLs motion, that is distinguished both by the non-finiteness of the variance because of $\alpha \in (1, 2)$, and by the no long-term dependence. The characteristics of the financial risk arising from the corresponding distributional law permit, again, the matching between the demand and the supply, but to a lower degree as compared to the "regular" phase. In fact, in the current case, the only source of attractiveness for investors who value lengthy horizons is the statistical self-similarity. In particular, notice that the values of the Hurst exponents, characterizing the "regular" and the "semi-regular" phases are within a limited range and, so, their impacts on the liquidity levels are quite similar for both phases. At least, no significant differences are apparent. To the contrary, the unpredictability of future returns due to the absence of some long-term dependence puts the "semi-regular" phase volatility in a higher risk class than does the unpredictability of the "regular" phase (however, *ex ceteris paribus*, both normal). Furthermore, the distributional properties of the underlying stochastic process put this PLs volatility in a higher risk class than the normal one.[26] Of course, this latter financial risk characteristic causes a lower participation of investors in the "semi-regular" foreign currency market than in the "regular" foreign currency market and, in particular, a lower participation of investors having long horizon lengths that are associated with highest risk. Because of this, in the corresponding "semi-regular" foreign currency market there are both a lower liquidity level, and a lower average investment horizon length than in the "regular" phase foreign currency one.

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25. Notice that, because of the trend due to long-term dependence, the standard deviation of the considered fBms provides an over-evaluation of the actual volatility of the corresponding foreign currency markets.

26. Recall that the tails of the PLs motions with $\alpha \in (0,2)$ decay slower than the fBm ones.

In order to explain such a higher risk level distinguishing the "semi-regular" phase, we can conjecture that the corresponding foreign currency markets are characterized by an irregular arrival of exogenous noise. Of course, this makes it difficult for investors to detect any trends that may exist in the fundamentals of the economy and thus may influence their ability to make rational decisions.

The "irregular" phase is associated with an undetectable stochastic process, that may be a suitable mixture of fBms and PLs motions, or which may belong to some proper PLs distribution sub-family. Although such lack of detection is possible, the (generic) identifiable characteristics of the corresponding distributional law (and, consequently, of the financial risk) are such as to prevent a simple matching between the demand and supply. In fact, in this "irregular" phase, volatility belongs to a risk class quite similar to the one that characterizes the "semi-regular" phase. Again, this causes primarily a lower participation of investors having long horizon lengths (who are associated with a higher level of risk) and, consequently, a lower liquidity level and a lower average investment horizon length than in the "regular" phase foreign currency markets. Moreover, the underlying stochastic process may or may not be characterized by the statistical self-similarity. In the first case, for the "irregular" phase, the corresponding Hurst exponent, H, is lower than that for the "regular" and "semi-regular" phases. It is simple to prove, under a reasonable assumption on a, that the proportionality factor a^{-H} is higher for these latter phases.[27] In the second case different horizon length investors do not evaluate investments in the same proportional way, and so their demands and supplies do not match.

In particular, in order to explain such a financial environment, we can conjecture that the corresponding foreign currency markets are characterized by the arrival of conflicting information. This causes very different and, often, incompatible behavior among the economic agents.

VII. Concluding Remarks

All the foreign currency markets studied in this article exhibit a Hurst exponent that is statistically different from 0.5 in the majority of the samples studied. Furthermore, it is also found that the Hurst exponent is

27. A simple proof of this claim may be obtained from the authors.

not fixed but it changes dynamically over time. The interpretation of these results is that the foreign currency returns follow either a fractional Brownian motion or a Pareto-Levy stable distribution. The key question is: what are the implications of such findings on the Efficient Market Hypothesis? Both in its original formulation and in the recent more sophisticated elaborations of the random walk hypothesis found in Campbell, Lo and MacKinlay (1997), the efficient market hypothesis is associated with returns that follow a Brownian motion with Hurst exponent equal to 0.5. Rogers (1997) has shown that a market where the asset returns follow a fractional Brownian motion cannot be efficient since there always exists an arbitrage strategy. Our approach has been to use the statistical evidence in this article to support the proposed Multi-Fractal Market Hypothesis. Needless to say, this extension of the traditional Efficient Market Hypothesis needs a further elaboration that goes beyond the general ideas we have offered in the previous sections. In particular, we need to develop theoretical explanations for both long-term positive and negative dependence as well as explanations for the transition of distributions from Brownian to fractally Brownian or Pareto-Levy stable.

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