Skewed Generalized Error Distribution of Financial Assets and Option Pricing

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This article provides a mathematical and empirical investigation of the reasons for the presence of skewness and kurtosis in financial data. The results indicate that this phenomenon is triggered by higher-order moment dependencies in the data, such as asymmetric and conditional volatility. Moreover, the article develops and tests successfully a skewed extension of the generalized error distribution (SGED), which is then used to model European call option prices. Under the standard assumptions of risk neutrality, normality of log-returns, and absence of arbitrage opportunities, the SGED model yields as special cases several well-known models for pricing options on stocks, stock indices, currencies, and currency futures. (JEL: C13, C22, G12, G13)

Keywords: asymmetric volatility; call option pricing; conditional heteroskedasticity; geometric Brownian motion; skewed GED

I. Introduction

Commonly used option pricing models, including that of Black-Scholes, are based on the assumption that prices of financial assets follow the geometric Brownian motion. A key assumption of the geometric Brownian motion is that the ratio of two consecutive prices does not

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depend on past prices. Furthermore, the natural logarithm of two consecutive prices, known as the log-return, is assumed to be normally distributed with the same annualized mean and variance over time.[1]

The geometric Brownian motion's assumptions imply that log-returns are identically and independently distributed (i.i.d.) normal variables, thus they exhibit no moment dependencies, such as asymmetric and conditional volatility. The presence of conditional volatility in the log-return series of financial assets is well documented in the GARCH literature, e.g., Akgiray (1989) and Ballie and Bollerslev (1989). The EGARCH literature further documents that volatility is larger during market downturns than market upturns (asymmetric volatility), e.g., Hsieh (1989), Nelson (1991), and Booth, Martikainen, and Tse (1997). As shown in this article, these dependencies are likely to trigger skewness and kurtosis in of log-return distributions.

The nature of option pricing necessitates the use of probability distributions which provide a good fit to the empirical distribution of log-returns accommodating skewness and kurtosis. These distributions are required to have all their moments defined, i.e., have a moment generating function and include the normal distribution as a special case. The latter assumption is consistent with the assumption that price changes follow a geometric Brownian motion.

This article provides a mathematical and empirical investigation of the reasons for the presence of skewness and kurtosis in financial data. To accommodate these properties, the article formally develops a skewed extension of the generalized error distribution (SGED) and tests its ability to model the empirical distribution of log-returns of several popular financial series. The SGED is then used to derive a pricing model for European call options that accounts for skewness and kurtosis. As shown in the article, the SGED model nests several well-known call option pricing models for stocks, stock indices, currencies, and currency futures.

The article is organized as follows. The next section presents a preliminary statistical analysis of the data. Section III investigates the reasons for the presence of skewness and kurtosis in the data. Section IV discusses the implications of skewness and kurtosis for the geometric

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1. The log-return is $y_t = \ln(P_t/P_{t-1}) = \ln(1+r_t)$, where r_t is the return for the period t–1 to t. The return r_t could be expressed in terms of y_t as $r_t = \exp(y_t) - 1$. Unlike returns, log-returns can be added across periods. For example, a monthly log-return is equal to the sum of daily log-returns during the month, i.e., $Y_n = \Sigma\, y_t$, where n is the number daily log-returns. The linear nature of log-returns simplifies considerably the computation of annualized means and variances of log-returns.

TABLE 1. Financial Series

No	Name	Symbol	OBS	Period of Data
1.	Boeing	BA	10,091	1/3/1975 - 12/31/2014
2.	IBM	IBM	10,091	1/3/1975 - 12/31/2014
3.	DJIA	$INDU	10,435	1/2/1975 - 12/31/2014
4.	S&P100	$OEX	8,042	3/6/1984 - 12/31/2014
5.	S&P500	$INX	10,435	1/2/1975 - 12/31/2014
6.	GBP/USD	GBP/USD	10,435	1/2/1975 - 12/31/2014
7.	JY/USD	JY/USD	7,564	1/3/1986 - 12/31/2014

Note: The stocks and currencies included in the above list are actively traded in U.S. financial markets. Options, futures, and other derivative assets written on them are very popular in the U.S. The currencies are expressed in terms of the U.S. dollar.

Brownian motion. Section V derives and applies the SGED to the data. Section VI presents the derivations for the SGED option pricing model. The last section offers the conclusions.

II. Data and Preliminary Findings

Table 1 lists the financial series and periods covered by the data. The data include daily, weekly, monthly and quarterly log-returns for two widely-held stocks (Boeing and IBM), three stock indices (Dow Jones Industrial Average (DJIA), S&P100, and S&P500), and two U.S. dollar exchange rates (British pound and Japanese yen). All of these assets have actively traded options. For example, call options on Boeing and IBM stocks and on the DJIA, S&P100, and S&P500 stock indices are actively traded on the Chicago Board of Options Exchange (CBOE). The data spans the period January 2, 1975 to December 31, 2014.

Log-returns at all frequencies are computed using the formula

$$y_t = 100 \times \ln\left(P_t / P_{t-1}\right), \tag{1}$$

where P_t is the price at the end of period t. For example, using monthly data, P_{t-1} and P_t and are the closing prices on the last trading day of two consecutive months t–1 and t. It can be readily shown that the additive property implies that a monthly (or quarterly) log-return is equal to the sum of all daily returns during the month (or quarter). As a consequence, statistics such as the mean and variance of lower

frequency data can be derived from those of higher frequency data.

Table 2 reports basic statistics of log-returns for all series examined, starting with the mean, the standard deviation, and the minimum and maximum values of log-returns in the first four rows. To allow comparison across the four data frequencies, the statistics are annualized using the formulas $\bar{y}/\Delta t$, $S/\sqrt{\Delta t}$, min/Δt, and max/Δt, where $\Delta t = 1/252$, 1/52, 1/12 and 1/4, for the daily, weekly, monthly and quarterly frequencies, respectively, where $\bar{y}$ is the simple arithmetic mean and S the sample standard deviation.

The annualized means for the stocks and stock indices range between 4.6552% and 14.0506% and the means for the two currencies between –0.9355% and –2.0902%. Note that the mean and standard deviation of log-returns for each series are quite similar across the four data frequencies examined; a result which is due to the additive property of log-returns and the linearity of the mean estimator. The similarity of standard deviations across the four data frequencies suggests that data of any frequency may be used to estimate the mean and volatility of asset log-price changes. In most cases, the standard deviations of log-returns are larger than their means, indicating high volatility. This phenomenon is confirmed by the large discrepancies between the minimum and maximum values of log-returns.

The next four rows give the statistics and t-values for standardized skewness and standardized kurtosis in excess of three, which is the standardized kurtosis for the normal distribution, computed using the equations

$$b_1 = \hat{m}_3 / \hat{m}_2^{3/2} \tag{2}$$

and

$$b_2 = \left(\hat{m}_4 / \hat{m}_2^2 \right) - 3, \tag{3}$$

where

$$\hat{m}_j = \sum (y_t - \bar{y})^j / T \tag{4}$$

is the estimate for the jth moment around the sample mean. Under the null hypothesis of normality, the two statistics are normally distributed with standard errors $SE(b_1) = \sqrt{6/T}$ and $SE(b_1) = \sqrt{6/T}$.

In general, the data series exhibit significant negative skewness across all four frequencies. The only exceptions are the daily and weekly log-returns for Boeing and the monthly log-returns for IBM. The excess kurtosis values are statistically significant in all cases implying

TABLE 2. Preliminary Statistics

Estimates	Daily	Weekly	Monthly	Quarterly
A. Boeing				
Mean	13.9601	13.9724	14.0506	13.9853
Standard Deviation	31.6945	31.7474	31.3298	32.6269
minimum	–4,886.1057	–1,227.4442	–519.4369	–253.0499
maximum	8,903.5831	2,035.2159	406.4927	186.1446
b_1	0.415	0.1017	–0.2593	–0.3717
	(17.02)**	(1.9)	(–2.32)*	(–1.91)
b_2	14.0881	4.7511	2.1454	1.2939
	(288.88)**	(44.3)**	(9.58)**	(3.33)**
BJ	83,739.36**	1,966.5**	97.23**	14.75**
Observations	10,091	2,087	479	159
B. IBM				
Mean	4.6647	4.6552	4.6967	5.1069
Standard Deviation	26.5701	27.2054	24.9627	26.0154
minimum	–6,574.2738	–1,887.1086	–360.7416	–187.3526
maximum	3,116.3503	1,196.0854	355.4328	153.1533
b_1	–0.3289	–0.3819	0.089	–0.2594
	(–13.49)**	(–7.12)**	(0.8)	(–1.34)
b_2	12.9657	7.8463	1.6506	1.4358
	(265.86)**	(73.17)**	(7.37)**	(3.7)**
BJ	70,865.24**	5,404.25**	55.01**	15.44**
Observations	10,091	2,087	479	159
C. DJIA				
Mean	8.4156	8.4132	8.4226	8.3164
Standard Deviation	17.1063	17.6489	15.4373	16.1252
minimum	–6,689.9406	–1,828.5294	–324.37	–123.3565
maximum	2,742.6787	681.0413	172.3855	84.7035
b_1	–1.4536	–1.4609	–0.7128	–0.7582
	(–60.62)**	(–27.25)**	(–6.37)**	(–3.9)**
b_2	40.2445	22.9967	3.7743	1.4467
	(839.16)**	(214.45)**	(16.86)**	(3.72)**
BJ	707,872.46**	46,729.92**	324.88**	29.1**
Observations	10,435	2,087	479	159
D. S&P100				
Mean	7.9908	8.0110	8.0155	7.8642
Standard Deviation	18.69	18.4228	16.008	16.8535
minimum	–6,183.0804	–1,743.8139	–289.2516	–118.9073
maximum	2,780.9704	621.661	168.4502	92.9781

(*Continued*)

TABLE 2. **(Continued)**

Estimates	Daily	Weekly	Monthly	Quarterly
D. S&P100				
b_1	−1.2466	−1.5973	−0.9063	−0.9047
	(−45.64)**	(−26.15)**	(−7.11)**	(−4.1)**
b_2	29.4299	21.1519	3.0421	1.7858
	(538.72)**	(173.14)**	(11.93)**	(4.04)**
BJ	292,305.48**	30,659.75**	192.8**	33.12**
Observations	8,042	1,608	369	123
E. S&P500				
Mean	8.5096	8.5151	8.5166	8.4174
Standard Deviation	17.3256	17.7345	15.4574	16.3197
minimum	−5,976.8276	−1,659.9106	−296.1221	−112.5027
maximum	2,859.8281	673.4488	175.3387	88.0366
b_1	−1.1473	−1.2177	−0.8325	−0.8071
	(−47.85)**	(−22.71)**	(−7.44)**	(−4.15)**
b_2	28.2369	16.7235	3.3502	1.331
	(588.79)**	(155.95)**	(14.97)**	(3.43)**
BJ	348,957.75**	24,835.95**	279.33**	29**
Observations	10,435	2,087	479	159
F. GBP/USD				
Mean	−1.026	−1.0322	−1.0069	−0.9355
Standard Deviation	9.6346	10.2107	10.4678	10.5994
minimum	−1,022.6542	−530.2511	−167.7294	−83.273
maximum	1,218.1852	330.4369	157.0867	59.9604
b_1	−0.0528	−0.469	−0.3306	−0.438
	(−2.2)*	(−8.75)**	(−2.95)**	(−2.25)*
b_2	4.0676	3.8873	2.097	1.3898
	(84.82)**	(36.25)**	(9.37)**	(3.58)**
BJ	7,198.56**	1,390.54**	96.49**	17.88**
Observations	10,435	2,087	479	159
G. JY/USD				
Mean	−1.7475	−1.7123	−1.7891	−2.0902
Standard Deviation	11.3237	11.5016	11.4819	12.4587
minimum	−2,003.7751	−691.8273	−199.5999	−75.3561
maximum	1,521.7646	328.4774	122.2159	67.7685
b_1	−0.4264	−0.6155	−0.342	−0.4069
	(−15.14)**	(−9.77)**	(−2.6)**	(−1.78)
b_2	6.9059	4.5649	1.6913	0.426
	(122.6)**	(36.24)**	(6.43)**	(0.93)
BJ	15,260.11**	1,409.18**	48.13**	4.04
Observations	7,564	1,513	347	115

(*Continued*)

TABLE 2. (Continued)

Note: Log-returns for the period t to $t-1$ are computed using $y_t = 100 \times \ln(P_t/P_{t-1})$, where P_t is the price of the stock or the value of the currency or index at time t. The mean, variance, minimum and maximum return are based on annualized returns computed using $y_t/\Delta t$, where $\Delta t = 1/252$, $1/52$, $1/12$, and $1/4$ for the daily, weekly, monthly and quarterly frequencies. The statistics for skewness and excess kurtosis are computed using $b_1 = m_3 / m_2^{3/2}$, $b_2 = (m_4/m_2^2) - 3$, where m_i is the sample the ith centered moment. Parentheses include t-values statistics. The Bera-Jarque statistic for testing normality, $BJ = (T/24)(4b_1^2 + b_2^2)$, is distributed as $\chi^2(2)$ with 2 degrees of freedom. Its critical value at the five-percent level is 5.99 and a the one-percent level is 9.21. *,** statistically significant at the five- and one-percent level of significance.

that the log-return are leptokurtic relative to the normal distribution.

The ninth row of table reports the Bera-Jarque statistics for testing the null hypothesis of normality, given by

$$BJ = \left(T/24\right)\left(4b_1^2 + b_2^2\right). \tag{5}$$

The *BJ* is asymptotically $\chi^2(2)$ with two degrees of freedom. Its critical values are 5.99 at the five-percent level and 9.21 at the one-percent level. The *BJ* statistics confirm the significance of skewness and/or leptokurtosis by rejecting the normality assumption in all series.

In summary, these preliminary results indicate that distributions of the log-return series are leptokurtosis and, in most cases, skewed. These findings are, however, based on the null hypothesis of normality and provide no information as to the parametric distribution of the data. Moreover, the statistics for skewness and kurtosis may be magnified by the presence of outliers.

III. Causes of Skewness and Kurtosis

The central limit theorem states that the distribution of the sum of i.i.d. random variables approaches the normal distribution as the number of variables summed-up increases. Provided that daily log-returns are i.i.d. random variables, the distributions of quarterly log-returns, each being the sum of about 63 non-overlapping daily log-returns, should be normal. However, this is not the case because of the fact that daily log-returns exhibit strong higher order moment dependencies such as, asymmetric volatility, conditional heteroskedasticity (volatility), or other dependencies.

Let $Z_n = z_1 + z_2 + \cdots + z_n$ be the sum of n non-overlapping random variables, where $z_t = (y_t - \mu)/\sigma$, and μ and σ are the mean and standard deviation of the log-return y_t. As such, the third and fourth moments of z_t, denoted by m_3 and m_4, are the standardized skewness and kurtosis of y_t. The expected value of Z_n is zero, i.e., $E(Z_n) = 0$. Moreover, under the assumption of i.i.d. log-returns $E(z_t z_s) = 0$, for $t \neq s$ and $\text{var}(Z_n) = n$.

The third and fourth centered moments of Z_n are

$$M_3 = EZ_n^3 = E\left(z_1 + z_2 + \cdots + z_n\right)^3$$

$$= \sum_t Ez_t^3 + 3\sum_{t \neq s}\sum Ez_t^2 z_s + \sum_{t \neq s \neq p}\sum\sum Ez_t z_s z_p \qquad (6)$$

and

$$M_4 = EZ_n^4 = E\left(z_1 + z_2 + \cdots + z_n\right)^4$$

$$= \sum_t Ez_t^4 + 4\sum_{t \neq s}\sum Ez_t^3 z_s + 3\sum_{t \neq s}\sum Ez_t^2 z_s^2$$

$$+3\sum_{t \neq s \neq p}\sum\sum Ez_t^2 z_s z_p + \sum_{t \neq s \neq p \neq r}\sum\sum\sum Ez_t z_s z_p z_r. \qquad (7)$$

Provided that z_t are i.i.d., for $t \neq s \neq p \neq r$,

$$Ez_t z_s z_p = 0, \qquad (8)$$

$$Ez_t^2 z_s = 0, \qquad (9)$$

$$Ez_t z_s z_p z_r = 0, \qquad (10)$$

$$Ez_t^2 z_s z_p = 0, \qquad (11)$$

$$Ez_t^2 z_s^2 = Ez_t^2 Ez_s^2 = 1, \qquad (12)$$

and

$$Ez_t^3 z_s = 0. \qquad (13)$$

Deviations of the above measures from their null values are evidence of the existence of higher order moment dependencies in the distribution of log-returns. Their substitutions into equations (6) and (7) give

$$M_3 = EZ_n^3 = n m_3, \tag{14}$$

and

$$M_4 = EZ_k^4 = n m_4 + 3n(n-1). \tag{15}$$

where $m_3 = Ez^3$ and $m_4 = Ez^4$ are respectively the standardized skewness and standardized kurtosis measures for z and y as well. The skewness and kurtosis for Z_n are

$$SK = \frac{EZ_n^3}{\mathrm{var}(Z_n)^{3/2}} = \frac{n m_3}{n^{3/2}} = \frac{m_3}{\sqrt{n}}, \tag{16}$$

and

$$KU = \frac{EZ_n^4}{\mathrm{var}(Z_n)^2} = \frac{n m_4 + 3n(n-1)}{n^2} = \frac{m_4}{n} + 3\left(1 - \frac{1}{n}\right). \tag{17}$$

It follows easily from the above equations that as $n \to \infty$ the skewness measure SK approaches the value of zero and the kurtosis KU the value of three which the standardized skewness and kurtosis values for the normal distribution.

Equations (6) and (7) provide the foundation for constructing test statistics for higher-order moment dependencies in financial data,

$$mz_1 = \frac{\sum \hat{z}_t^2 \hat{z}_{t-j}}{T} \text{ and } \mathrm{var}(mz_1) = \frac{Ez_t^4}{T} \tag{18a}$$

$$mz_2 = \frac{\sum \hat{z}_t \hat{z}_{t-1} \hat{z}_{t-2}}{T} \text{ and } \mathrm{var}(mz_2) = \frac{1}{T}, \tag{18b}$$

$$mz_3 = \frac{\sum \hat{z}_t^3 \hat{z}_{t-1}}{T} \text{ and } \mathrm{var}(mz_3) = \frac{Ez_t^6}{T}, \tag{18c}$$

$$mz_4 = \frac{\sum\left(\hat{z}_t^2 \hat{z}_{t-j}^2 - 1\right)}{T} \text{ and } \mathrm{var}\left(mz_4\right) = \frac{\left(Ez_t^4\right)^2}{T}, \qquad (18d)$$

$$mz_5 = \frac{\sum \hat{z}_t^2 \hat{z}_{t-1}\hat{z}_{t-2}}{T} \text{ and } \mathrm{var}\left(mz_5\right) = \frac{Ez_t^4}{T}, \qquad (18e)$$

and

$$mz_6 = \frac{\sum \hat{z}_t\hat{z}_{t-1}\hat{z}_{t-2}\hat{z}_{t-3}}{T} \text{ and } \mathrm{var}\left(mz_6\right) = \frac{1}{T}, \qquad (18f)$$

where $\hat{z}_t = \left(y_t - \overline{y}\right)/S$, $\overline{y}$ and S are respectively the sample mean and standard deviation of y_t, and T is the sample size and $j = 1, 2$ and 3. The moments Ez_t^4 and Ez_t^6 are estimated using respectively the statistics

$$\hat{m}_4 = \sum \hat{z}_t^4 \big/ T \qquad (19)$$

and

$$\hat{m}_6 = \sum \hat{z}_t^6 \big/ T. \qquad (20)$$

The variances of the statistics above are derived under the null hypothesis that the sequence z_t includes i.i.d. random variables, which implies that all the statistics in (18) have a zero mean. Note that there is a large number of statistics generated by (6) and (7) Thus, the above test statistics are not exhaustive of all higher-order moment dependencies in the data.

Table 3 presents the statistics given by (18a-f) along with their t-values for all log-return series and frequencies. Statistically significant monthly and occasionally in quarterly log-returns of all seven series. These relationships weaken with aggregation. Many of the dependencies documented on table 3 are consistent with earlier results on the stochastic nature of log-returns of financial assets. This literature reports that volatility is significantly clustered (conditional heteroskedasticity), or that large log-returns are likely to be followed by large log-returns of lesser magnitude, but undetermined sign. The latter implies

$$Ez_t^2 z_s^2 \neq Ez_t^2 Ez_s^2, \text{for } t \neq s.$$

The statistics based on $z_t^2 z_s^2$ for $s = t-1, t-2$ and $t-3$, test for conditional heteroskedasticity in the data, the presence of which is revealed in all

TABLE 3.   Higher Order Moment Dependencies

Estimates	Daily	Weekly	Monthly	Quarterly
A. Boeing				
$z_t^2\, z_{t-1}$	−0.0965	−0.21	−0.172	0.0307
	(−2.35)*	(−3.45)**	(−1.66)	(0.19)
$z_t^2\, z_{t-2}$	−0.1377	−0.1126	0.0999	−0.3844
	(−3.35)**	(−1.85)	(0.96)	(−2.35)*
$z_t^2\, z_{t-3}$	−0.1573	−0.0943	−0.2778	−0.308
	(−3.82)**	(−1.55)	(−2.68)**	(−1.88)
$z_t\, z_{t-1}\, z_{t-2}$	−0.0288	0.0653	0.0562	−0.0879
	(−0)	(0)	(0)	(−0.01)
$z_t^3\, z_{t-1}$	0.0031	0.0054	1.1194	0.5817
	(0.01)	(0.01)	(3.14)**	(1.16)
$z_t^2\, z_{t-1}^2 -1$	1.1287	0.5898	0.4148	−0.0166
	(6.64)**	(3.48)**	(1.77)	(−0.05)
$z_t^2\, z_{t-2}^2 -1$	0.725	0.4735	0.3771	0.1155
	(4.26)**	(2.79)**	(1.61)	(0.34)
$z_t^2\, z_{t-3}^2 -1$	0.806	0.2809	0.8353	0.0174
	(4.74)**	(1.66)	(3.56)**	(0.05)
$z_t^2\, z_{t-1}\, z_{t-2}$	0.1832	0.0646	−0.0966	0.2156
	(4.45)**	(1.06)	(−0.93)	(1.32)
$z_t\, z_{t-1}\, z_{t-2}\, z_{t-3}$	0.0523	−0.0206	−0.1298	0.1138
	(5.26)**	(−0.94)	(−2.84)**	(1.43)
B. IBM				
$z_t^2\, z_{t-1}$	−0.161	−0.2456	−0.2562	−0.3069
	(−4.05)**	(−3.41)**	(−2.6)**	(−1.84)
$z_t^2\, z_{t-2}$	−0.1381	−0.0966	−0.0944	0.2382
	(−3.47)**	(−1.34)	(−0.96)	(1.43)
$z_t^2\, z_{t-3}$	−0.0995	−0.054	−0.122	−0.0712
	(−2.5)*	(−0.75)	(−1.24)	(−0.43)
$z_t\, z_{t-1}\, z_{t-2}$	−0.0151	0.0254	0.0592	0.1169
	(−0)	(0)	(0)	(0.01)
$z_t^3\, z_{t-1}$	0.461	0.2771	−0.2201	0.5829
	(1.08)	(0.56)	(−0.72)	(1.2)
$z_t^2\, z_{t-1}^2 -1$	1.9619	0.7696	0.4041	0.1625
	(12.34)**	(3.24)**	(1.91)	(0.46)
$z_t^2\, z_{t-2}^2 -1$	1.153	0.7991	0.3011	0.6947
	(7.26)**	(3.37)**	(1.42)	(1.99)*
$z_t^2\, z_{t-3}^2 -1$	0.7974	0.5243	0.5359	0.1366
	(5.02)**	(2.21)*	(2.53)*	(0.39)
$z_t^2\, z_{t-1}\, z_{t-2}$	0.089	−0.2333	0.0816	−0.4637
	(2.24)*	(−3.24)**	(0.83)	(−2.78)**
$z_t\, z_{t-1}\, z_{t-2}\, z_{t-3}$	0.0323	0.0485	−0.0172	−0.2355
	(3.25)**	(2.22)*	(−0.38)	(−2.97)**

(*Continued*)

TABLE 3. (Continued)

Estimates	Daily	Weekly	Monthly	Quarterly
C. DJIA				
$z_t^2\, z_{t-1}$	–0.4917	–0.5419	–0.442	–0.0555
	(–7.64)**	(–4.86)**	(–3.72)**	(–0.33)
$z_t^2\, z_{t-2}$	–0.5663	–0.2506	–0.1045	–0.0669
	(–8.8)**	(–2.25)*	(–0.88)	(–0.4)
$z_t^2\, z_{t-3}$	–0.3585	–0.148	–0.026	0.1786
	(–5.57)**	(–1.33)	(–0.22)	(1.07)
$z_t\, z_{t-1}\, z_{t-2}$	–0.0423	0.0463	0.1188	–0.1567
	(–0)	(0)	(0.01)	(–0.01)
$z_t^3\, z_{t-1}$	5.6172	3.3654	0.445	–0.2178
	(4.07)**	(2.32)*	(0.8)	(–0.43)
$z_t^2\, z_{t-1}^2 -1$	4.4113	1.9363	0.8995	0.144
	(10.42)**	(3.4)**	(2.91)**	(0.41)
$z_t^2\, z_{t-2}^2 -1$	7.4108	2.8301	0.2878	0.0195
	(17.51)**	(4.98)**	(0.93)	(0.06)
$z_t^2\, z_{t-3}^2 -1$	3.6972	1.4791	0.4151	0.4481
	(8.73)**	(2.6)**	(1.34)	(1.28)
$z_t^2\, z_{t-1}\, z_{t-2}$	0.0131	–0.5253	0.2275	0.1238
	(0.2)	(–4.71)**	(1.91)	(0.74)
$z_t\, z_{t-1}\, z_{t-2}\, z_{t-3}$	0.9713	0.1527	–0.1127	–0.0022
	(99.22)**	(6.98)**	(–2.47)*	(–0.03)
D. S&P100				
$z_t^2\, z_{t-1}$	–0.5776	–0.5289	–0.618	–0.3163
	(–9.1)**	(–4.32)**	(–4.84)**	(–1.61)
$z_t^2\, z_{t-2}$	–0.5553	–0.3253	–0.2132	–0.1624
	(–8.75)**	(–2.66)**	(–1.67)	(–0.83)
$z_t^2\, z_{t-3}$	–0.3382	–0.202	–0.1339	0.1408
	(–5.33)**	(–1.65)	(–1.05)	(0.72)
$z_t\, z_{t-1}\, z_{t-2}$	–0.0295	0.0522	0.2118	–0.1613
	(–0)	(0)	(0.01)	(–0.01)
$z_t^3\, z_{t-1}$	4.2016	2.9331	0.7676	–0.1474
	(3.84)**	(2.03)*	(1.52)	(–0.26)
$z_t^2\, z_{t-1}^2 -1$	5.4177	2.0137	1.5671	0.2367
	(14.98)**	(3.35)**	(5)**	(0.55)
$z_t^2\, z_{t-2}^2 -1$	5.8484	3.0787	0.6404	0.1138
	(16.17)**	(5.11)**	(2.04)*	(0.27)
$z_t^2\, z_{t-3}^2 -1$	3.0715	1.6822	0.5724	0.7665
	(8.49)**	(2.79)**	(1.82)	(1.79)
$z_t^2\, z_{t-1}\, z_{t-2}$	0.2717	–0.6468	0.4206	0.1216
	(4.28)**	(–5.28)**	(3.29)**	(0.62)
$z_t\, z_{t-1}\, z_{t-2}\, z_{t-3}$	1.0070	0.1776	–0.252	–0.0646
	(90.3)**	(7.12)**	(–4.84)**	(–0.72)

(*Continued*)

TABLE 3. (Continued)

Estimates	Daily	Weekly	Monthly	Quarterly
E. S&P500				
$z_t^2\, z_{t-1}$	−0.5032	−0.501	−0.5956	−0.2369
	(−9.2)**	(−5.15)**	(−5.18)**	(−1.44)
$z_t^2\, z_{t-2}$	−0.5066	−0.2919	−0.1784	−0.1603
	(−9.26)**	(−3)**	(−1.55)	(−0.97)
$z_t^2\, z_{t-3}$	−0.3027	−0.2132	−0.1563	0.0799
	(−5.53)**	(−2.19)*	(−1.36)	(0.49)
$z_t\, z_{t-1}\, z_{t-2}$	0.0002	0.0604	0.1752	−0.1974
	(0)	(0)	(0.01)	(−0.02)
$z_t^3\, z_{t-1}$	4.0284	2.0241	0.9435	0.0972
	(4.18)**	(1.86)	(1.97)*	(0.21)
$z_t^2\, z_{t-1}^2 - 1$	4.1478	1.8058	1.5873	0.3118
	(13.57)**	(4.18)**	(5.48)**	(0.91)
$z_t^2\, z_{t-2}^2 - 1$	6.1953	2.6703	0.7719	0.0942
	(20.26)**	(6.19)**	(2.67)**	(0.28)
$z_t^2\, z_{t-3}^2 - 1$	3.3222	1.487	0.732	0.4534
	(10.87)**	(3.45)**	(2.53)*	(1.33)
$z_t^2\, z_{t-1}\, z_{t-2}$	0.1349	−0.4591	0.5022	0.23
	(2.46)*	(−4.72)**	(4.37)**	(1.4)
$z_t\, z_{t-1}\, z_{t-2}\, z_{t-3}$	0.7692	−0.0431	−0.3686	−0.0313
	(78.57)**	(−1.97)*	(−8.07)**	(−0.39)
F. GBP/USD				
$z_t^2\, z_{t-1}$	−0.1124	−0.0536	−0.272	−0.3229
	(−4.32)**	(−0.93)	(−2.64)**	(−1.95)
$z_t^2\, z_{t-2}$	−0.0695	0.0481	−0.0254	−0.0394
	(−2.67)**	(0.84)	(−0.25)	(−0.24)
$z_t^2\, z_{t-3}$	−0.0379	−0.0504	0.0235	−0.0337
	(−1.46)	(−0.88)	(0.23)	(−0.2)
$z_t\, z_{t-1}\, z_{t-2}$	−0.0174	−0.0403	0.0337	0.0896
	(−0)	(−0)	(0)	(0.01)
$z_t^3\, z_{t-1}$	0.8804	0.733	0.5166	1.1708
	(7.25)**	(2.57)*	(1.44)	(2.41)*
$z_t^2\, z_{t-1}^2 - 1$	0.7569	0.7717	0.8117	0.2926
	(10.94)**	(5.12)**	(3.49)**	(0.85)
$z_t^2\, z_{t-2}^2 - 1$	0.6582	0.4755	0.2927	0.6245
	(9.51)**	(3.16)**	(1.26)	(1.81)
$z_t^2\, z_{t-3}^2 - 1$	0.7835	0.9536	0.1413	0.1748
	(11.33)**	(6.33)**	(0.61)	(0.51)
$z_t^2\, z_{t-1}\, z_{t-2}$	0.2365	−0.0423	0.0554	−0.185
	(9.09)**	(−0.74)	(0.54)	(−1.12)
$z_t\, z_{t-1}\, z_{t-2}\, z_{t-3}$	0.0404	0.006	0.0296	−0.0298
	(4.13)**	(0.27)	(0.65)	(−0.38)

(*Continued*)

TABLE 3.   (Continued)

Estimates	Daily	Weekly	Monthly	Quarterly
G. JY/USD				
$z_t^2\, z_{t-1}$	–0.2592	–0.1378	–0.0922	–0.036
	(–7.16)**	(–1.95)	(–0.79)	(–0.21)
$z_t^2\, z_{t-2}$	–0.1713	–0.0025	–0.2424	–0.0011
	(–4.73)**	(–0.03)	(–2.09)*	(–0.01)
$z_t^2\, z_{t-3}$	–0.1118	–0.0613	0.138	–0.1684
	(–3.09)**	(–0.87)	(1.19)	(–0.98)
$z_t\, z_{t-1}\, z_{t-2}$	0.0416	0.0063	0.1156	0.0641
	(0)	(0)	(0.01)	(0.01)
$z_t^3\, z_{t-1}$	0.2782	–0.2931	0.039	–0.0113
	(1.14)	(–0.69)	(0.09)	(–0.03)
$z_t^2\, z_{t-1}^2 - 1$	1.2951	0.6842	0.1803	–0.3811
	(11.37)**	(3.52)**	(0.72)	(–1.2)
$z_t^2\, z_{t-2}^2 - 1$	0.8722	0.4493	0.3293	0.0697
	(7.66)**	(2.31)*	(1.31)	(0.22)
$z_t^2\, z_{t-3}^2 - 1$	0.3597	0.3841	0.0544	0.0727
	(3.16)**	(1.98)*	(0.22)	(0.23)
$z_t^2\, z_{t-1}\, z_{t-2}$	0.1944	–0.0373	–0.0318	0.0103
	(5.37)**	(–0.53)	(–0.27)	(0.06)
$z_t\, z_{t-1}\, z_{t-2}\, z_{t-3}$	0.0406	–0.0956	–0.0083	–0.0345
	(3.53)**	(–3.72)**	(–0.16)	(–0.37)
Observations	7,564	1,513	347	115

Note: The random variable $z = (y-\mu)/\sigma$ is standardized, i.e., ita has mean ero and standard deviation one. The table presents the mean values of the constructs over the entire sampling period across the four data frequencies. The parameter μ and σ are replaced by the simple arithmetic mean and sample standard deviation. Parentheses include the t-values for the estimates. *,** implies statistically significant higher order moment dependencies at the five- and one-percent level of significance.

daily and weekly log-returns. Similar relationships for stocks, stock indices, and exchange rates are documented by a number of authors including Akgiray (1989) and Baillie and Bollerslev (1989).

The EGARCH literature reports that stock market volatility is larger during market downturns than market upturns, e.g., Nelson (1991), and Booth, Martikainen, and Tse (1997), Chiang et al. (2013), Deleze and Houssain (2014), Ferguson et al (2015), Gulati et al. (2013). This finding implies that

$$E\left(z_t^2 \,|\, z_s < 0\right) > E\left(z_t^2 \,|\, z_s > 0\right),$$

or that

$$Ez_t^2 z_s \neq 0, \text{ for } t > s.$$

Statistics based on the product $z_t^2 z_s$, for $s = t-1$, $t-2$ and $t-3$, test for the presence of asymmetric volatility in the data. The statistics reported in table 3 reveal the presence of asymmetric volatility in almost all daily, weekly and monthly log-return series. The negative statistics imply that volatility is higher in a bearish market than in a bullish one.

The remaining statistics test for more complex higher-order moment dependencies (or non-linearities) that may cause skewness and kurtosis in lower frequency data. The results document such dependencies at the daily, weekly and monthly data frequencies. These higher order-moment dependencies are responsible for skewness and kurtosis in financial log-return data; see equations (6) and (7).

IV. Geometric Brownian Motion

Various option pricing models, including that of Black-Scholes, are based on the assumption that prices of financial assets follow the geometric Brownian motion. As a result, the ratio of two consecutive prices does not depend on past prices and its log-value (log-returns) and is normally distributed. Therefore, the ratio of two consecutive prices is log-normally distributed. The mathematical relation between log-returns ($\Delta \ln P_t$) and geometric returns ($\Delta P_t / P_t$) is as follows:

$$\Delta \ln P_t = \ln\left(\frac{P_{t+\Delta t}}{P_t}\right) = \ln\left(1 + \frac{\Delta P_t}{P_t}\right). \tag{21}$$

Let $\mu_a \Delta t$ and $\sigma_a^2 \Delta t$ be the mean and variance of $\Delta \ln P_t$, where μ_a and σ_a^2 are annualized measures and Δt is the length of time between two consecutive prices expressed as a fraction of the year, i.e., for daily log-returns $\Delta t = 1/252$. Consider the transformation

$$z \equiv \frac{\Delta \ln P_t - \mu_a \Delta t}{\sigma_a \sqrt{\Delta t}}. \tag{22}$$

The log-return can be written as

$$\Delta \ln P_t = \mu_a \Delta t + \sigma_a \sqrt{\Delta t}\, z, \tag{23}$$

where z is a standardized random variable. Under normality

$$\lim_{\Delta t \to \infty} \Delta \ln P_t = d \ln P_t = \mu_a + \sigma_a dz \tag{24}$$

where $dz = z\sqrt{dt}$ is a Wiener process. Equation (22) can be rewritten in terms of geometric returns as

$$1 + \Delta P_t / P_t = \exp\left(\mu_a \Delta t + \sigma_a \sqrt{\Delta t}\, z\right) \tag{25}$$

Its moment function is [2]

$$E\left(1 + \Delta P_t / P_t\right)^r = \exp\left(\left(r\mu_a + \tfrac{1}{2} r^2 \sigma_a^2\right)\Delta t\right). \tag{26}$$

It follows from the above that [3]

$$E\left(\Delta P_t / P_t\right) = \exp\left(\left(\mu_a + \tfrac{1}{2}\sigma_a^2\right)\Delta t\right) - 1 \approx \left(\mu_a + \tfrac{1}{2}\sigma_a^2\right)\Delta t \tag{27}$$

and

$$\mathrm{var}\left(\Delta P_t / P_t\right) = E\left(1 + \Delta P_t / P_t\right)^2 - \left(E\left(1 + \Delta P_t / P_t\right)\right)^2$$

$$= \exp\left(\left(2\mu_a + 2\sigma_a^2\right)\Delta t\right) - \exp\left(\left(2\mu_a + \sigma_a^2\right)\Delta t\right) \approx \sigma_a^2 \Delta t. \tag{28}$$

Along the same lines as with equation (22)

$$\Delta P_t / P_t = \left(\mu_a + \tfrac{1}{2}\sigma_a^2\right)\Delta t + \sigma_a \sqrt{\Delta t}\, z \tag{29}$$

where z is a standardized log-normal random variable. As $\Delta t \to 0$, the above equation yields that of the geometric Brownian motion

$$dP_t = \left(\mu_a + \tfrac{1}{2}\sigma_a^2\right)P_t\, dt + \sigma_a P_t\, dz, \tag{30}$$

where $\left(\mu_a + \tfrac{1}{2}\sigma_a^2\right)P_t$ and $\sigma_a P_t$ are the instantaneous drift the standard

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2. The moment function of $x = e^y$, where y is a normal with mean μ and variance σ^2, is obtained from the moment generating function of y, i.e., $Ex^r = Ee^{ry} = \exp\left(r\mu + \tfrac{1}{2}r^2\sigma^2\right)$.

3. $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$, where $x = \left(r\mu_a + \tfrac{1}{2}r^2\sigma_a^2\right)\Delta t$. Note that the terms of the expansion of order two or greater vanish as the interval Δt becomes short.

deviation of the process.

Ross (1999) shows that the GBM could also be obtained as a limit of a simpler process. For example, consider the ratio of two consecutive prices P_n/P_0. The logarithm of the ratio

$$Y_n = \ln\left(P_n/P_0\right) = \sum_{t=1}^{n} \ln\left(P_t/P_{t-1}\right) \qquad (31)$$

could be broken down into the sum of n higher frequency log-returns corresponding to period Δt and having mean $\mu_a \Delta t$ and variance $\sigma_a^2 \Delta t$. Under the assumption of i.i.d. log-returns, the log-return Y_n has a mean $E(Y_n) = \mu_a \tau$ and a variance $\text{var}(Y_n) = \sigma_a^2 \tau$, where $\tau = n \Delta t$ is the time between prices P_0 and P_n. For a large n, the central limit theorem states that Y_n will be approximately normal, i.e.,

$$P\left(\frac{Y_n - \mu_a \tau}{\sigma_a \sqrt{\tau}} \leq z\right) \approx \Phi(z), \qquad (32)$$

with the approximation becoming exact as $n \to \infty$.

The i.i.d. assumption for P_t/P_{t-1} causes the P_n/P_0 ratio to behave in the limit as a GBM. As shown in the previous section, financial data exhibit strong higher-order moment dependencies, such as asymmetric volatility and conditional heteroskedasticity, which prevent lower frequency log-returns from converging in probability to the normal distribution. These complex dependencies are responsible for the presence of skewness and kurtosis in the data. In this respect, the stochastic process governing price changes is complex and extremely difficult to approximate.

V. Skewed Generalized Error Distribution

Introduced by Subbotin (1923), the GED includes as special cases the Laplace, normal, and uniform distributions, e.g., Johnson, Kotz, and Balakrishnan (1995). The GED is used by Box and Tiao (1962) to model prior densities in Bayesian estimation, Nelson (1991) to model the distribution of stock market returns, and Hsieh (1989) to model the distribution of exchange rates.

The GED is nested in the generalized t distribution introduced by

McDonald and Newey (1988) and its skewed extension introduced by Theodossiou (1998); see also McDonald and Nelson (1989), Butler et al. (1990) and Theodossiou and Savva (2015). Unlike the generalized t distribution, all the moments and the moment generating function of GED exist. Thus, the moments of exponential transformations of GED random variables, needed to price options, can be evaluated.

In this section, the generalized error distribution (GED) is extented to accommodate skewness and leptokurtosis. Its ability to model the empirical distribution of log-returns of financial assets is also evaluated.

A. Skewed GED

The probability density function for the non-centered SGED is (see appendix A for the derivations)

$$f_y = \frac{k^{1-\frac{1}{k}}}{2\varphi} \Gamma\left(\frac{1}{k}\right)^{-1} \exp\left(-\frac{1}{k} \frac{|u|^k}{\left(1+sgn(u)\lambda\right)^k \varphi^k}\right) \tag{33}$$

where $u = y - m$, m is the mode of the random variable y, φ is a scaling constant related to the standard deviation of y, λ is a skewness parameter, k is a kurtosis parameter, sgn is the sign function taking the value of -1 for $u < 0$ and 1 for $u > 0$, and

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

is the gamma function. The kurtosis parameter k controls the tails and peakness of the distribution. The skewness parameter λ $(-1 \le \lambda \le 1)$ controls the rate of descent of the density around m. In the case of positive skewness $(\lambda > 0)$ the density function is skewed to the right and vice versa.

Its moment function is

$$M_{u,r} = Eu^r = A_r \phi^r \tag{34}$$

where

$$A_r = \frac{1}{2}\left[(-1)^r (1-\lambda)^{r+1} + (1+\lambda)^{r+1}\right]G_r, \tag{35}$$

$$G_r = k^{\frac{r}{k}} \Gamma\left(\frac{r+1}{k}\right) \Gamma\left(\frac{1}{k}\right)^{-1}. \tag{36}$$

The resulting expected value and variance of y are

$$\mu = E(y) = m + E(u) = m + A_1\varphi \tag{37}$$

and

$$\sigma^2 = Eu^2 - (Eu)^2 = \left(A_2 - A_1^2\right)\varphi^2 \tag{38}$$

where

$$A_1 = 2\lambda\, k^{\frac{1}{k}} \Gamma\left(\frac{2}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1} \tag{39}$$

and

$$A_2 = \left(1+3\lambda^2\right)k^{\frac{2}{k}} \Gamma\left(\frac{3}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1}. \tag{40}$$

The Pearson's skewness, see Stuart and Ord (1994), p. 108, is

$$\delta = (\mu - m)/\sigma = A_1 / \sqrt{A_2 - A_1^2}. \tag{41}$$

The standardized skewness of y is

$$SK = \frac{A_3 - 3A_2 A_1 + 2A_1^3}{\left(A_2 - A_1^2\right)^{3/2}} \tag{42}$$

and

$$A_3 = 2\lambda\left(1+\lambda^2\right)k^{\frac{3}{k}} \Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1}. \tag{43}$$

The standardized kurtosis is

$$KU = \frac{A_4 - 4A_3 A_1 + 6A_2 A_1^2 - 3A_1^4}{\left(A_2 - A_1^2\right)^2} \tag{44}$$

and

$$A_4 = \left(1+10\lambda^2 + 5\lambda^4\right)k^{\frac{4}{k}} \Gamma\left(\frac{5}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1}. \tag{45}$$

The SGED gives 1) for $\lambda = 0$ the Subbotin's or the power exponential distribution, 2) for $\lambda = 0$ and $k = 1$ the Laplace or double exponential
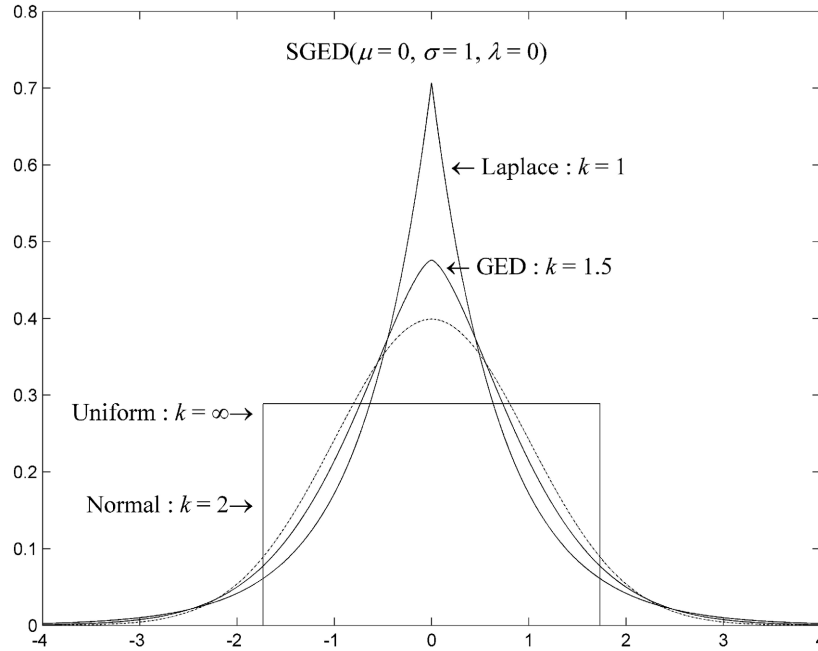
FIGURE 1.— Symmetric GED, Special Cases

distribution, 3) for $k = 2$ the skewed normal, 4) for $\lambda = 0$ and $k = 2$ the normal distribution, 5) for $\lambda = 1$ and $k = 1$ the exponential distribution and 6) for $\lambda = 0$ and $k = \infty$ the uniform distribution. Figure 1 presents a graphical illustration of the GED for a selected combination of the parameters k. Note that as k gets smaller, the tails of the distribution become flatter and the center becomes more peaked.

The constants φ and m can be expressed in terms of σ and μ as

$$\varphi = \varphi(\sigma, \lambda, k) = \gamma \sigma \tag{46}$$

and

$$m = m(\mu, \sigma, \lambda, k) = \mu - \delta \sigma \tag{47}$$

where

$$\gamma = 1 / \sqrt{A_2 - A_1^2} \tag{48}$$

A_1, A_2 and δ (Pearson's skewness) are as defined previously. The direct substitution of the above equations into equation (33) gives the centered

TABLE 4. Skewness and Kurtosis Relative to *k* and λ

A. Skewness

k / λ	0.05	0.15	0.25	0.5	0.75	0.85	1
0.8	0.291	0.849	1.342	2.163	2.473	2.508	2.511
0.9	0.246	0.719	1.141	1.870	2.170	2.211	2.224
1.0	0.212	0.620	0.989	1.644	1.935	1.979	2.000
1.1	0.185	0.543	0.869	1.464	1.745	1.792	1.819
1.2	0.164	0.482	0.773	1.317	1.587	1.637	1.669
1.3	0.146	0.431	0.694	1.194	1.455	1.506	1.542
1.4	0.132	0.389	0.628	1.090	1.341	1.393	1.432
1.5	0.120	0.354	0.571	1.001	1.243	1.295	1.337
1.6	0.109	0.323	0.523	0.923	1.157	1.209	1.253
1.7	0.100	0.297	0.482	0.856	1.080	1.132	1.178
1.8	0.093	0.274	0.445	0.796	1.012	1.064	1.111
1.9	0.086	0.254	0.413	0.742	0.951	1.002	1.050
2.0	0.080	0.236	0.385	0.695	0.896	0.946	0.995
5.0	0.020	0.060	0.099	0.190	0.268	0.294	0.328
10.0	0.006	0.019	0.031	0.061	0.090	0.100	0.115
∞	0.000	0.000	0.000	0.000	0.000	0.000	0.000

B. Kurtosis

k / λ	0.05	0.15	0.25	0.5	0.75	0.85	1
0.8	8.622	9.052	9.796	11.897	13.030	13.163	13.173
0.9	7.066	7.373	7.912	9.511	10.485	10.637	10.671
1.0	6.030	6.258	6.664	7.920	8.765	8.922	9.000
1.1	5.299	5.475	5.790	6.801	7.539	7.695	7.799
1.2	4.761	4.900	5.150	5.980	6.630	6.781	6.900
1.3	4.351	4.462	4.666	5.357	5.934	6.079	6.205
1.4	4.030	4.121	4.289	4.873	5.387	5.525	5.655
1.5	3.772	3.847	3.988	4.487	4.948	5.079	5.210
1.6	3.561	3.625	3.744	4.175	4.590	4.714	4.844
1.7	3.386	3.440	3.542	3.918	4.293	4.410	4.539
1.8	3.238	3.285	3.373	3.703	4.044	4.154	4.280
1.9	3.113	3.153	3.230	3.522	3.833	3.937	4.060
2	3.005	3.040	3.107	3.367	3.652	3.750	3.869
5	2.071	2.075	2.083	2.120	2.176	2.203	2.246
10	1.884	1.886	1.888	1.899	1.917	1.927	1.942
∞	1.800	1.800	1.800	1.800	1.800	1.800	1.800

Note: The above skewness and kurtosis values are computed using equations (35) and (37). Negative values of λ generate negative skewness values of equal magnitude as the corresponding positive values.
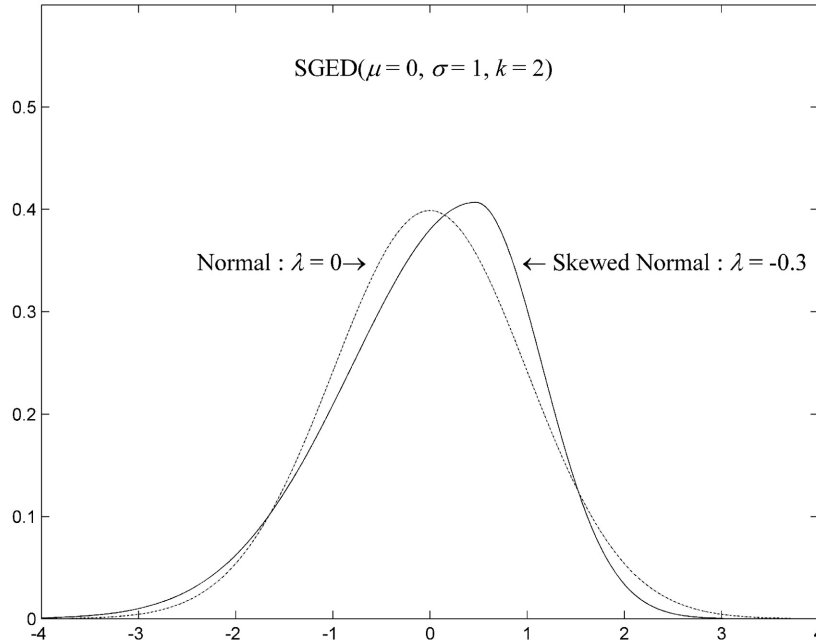
FIGURE 2.— Symmetric and Negatively Skewed Normal Distributions

version of the SGED below

$$f_y = \frac{k^{1-\frac{1}{k}}}{2\gamma\sigma}\Gamma\left(\frac{1}{k}\right)^{-1}\exp\left(-\frac{1}{k}\frac{\left|y-\mu+\delta\sigma\right|^k}{\left(1+sgn(y-\mu+\delta\sigma)\lambda\right)^k\gamma^k\sigma^k}\right). \quad (49)$$

Table 4 presents the skewness and kurtosis values for various combinations of k and λ. The results provide a better understanding of the relationship between k and λ and the shape of the SGED. It appears from the table that smaller values of k generate larger values for the kurtosis; i.e., more leptokurtic density functions. Moreover, larger positive values of λ result in larger positive values for both skewness and kurtosis. On the other hand, negative values of λ generate identical tables for skewness and kurtosis; however, all skewness values are negative. Figure 2 shows the impact of skewness on the normal density function.

B. SGED Results

The SGED parameters are obtained by maximizing the log-likelihood function of log-returns y_t, for $t = 1, 2, ..., T$,

$$\max_{\vartheta} L\left(\vartheta \middle| y_t\right) = \sum_{t=1}^{T} \ln f_y\left(\vartheta \middle| y_t\right) \tag{50}$$

with respect to the parameter vector $\vartheta = \left[m, \phi, g, k\right]'$, where

$$\varphi\left(\phi\right) = e^{\phi} \tag{51}$$

and

$$\lambda\left(g\right) = \frac{2}{1 + e^{-g}} - 1. \tag{52}$$

Both transformations are monotonic functions, therefore there is an one to one relationship between the parameters and their respective transformed values. Moreover, note that the range of φ and g is the real line and as such the overshooting problems associated with the constraints imposed on the parameters φ and λ, i.e., $\varphi > 0$ and $|\lambda| \leq 1$ is avoided. The non-centered rather than the centered is preferred because of its simplicity

For practical purposes in estimation, the likelihood is specified using the simplified non-centered rather than the centered probability specification. The maximization of the sample log-likelihood function is accomplished using an iterative procedure based on the Berndt, Hall, Hall and Hausman (1974) algorithm. That is, during each iteration, the values for θ are updated using the equation

$$\vartheta^{i+1} = \vartheta^{i} + Q\left(\vartheta^{i}\right)^{-1} S\left(\vartheta^{i}\right) \tag{53}$$

where

$$S\left(\vartheta^{i}\right) = \frac{\partial L\left(\vartheta^{i}\right)}{\partial \vartheta} = \sum_{t=1}^{T} \frac{\partial L_t\left(\vartheta^{i}\right)}{\partial \vartheta} \tag{54}$$

and

$$Q\left(\vartheta^{i}\right) = \sum_{t=1}^{T} \frac{\partial L_t\left(\vartheta^{i}\right)}{\partial \vartheta} \frac{\partial L_t\left(\vartheta^{i}\right)}{\partial \vartheta} \tag{55}$$

denote the first partial derivatives of the sample log-likelihood and the sum of their cross-products, respectively. These partial derivatives include psi (digamma) functions, computed using numerical techniques. To avoid complications associated with discontinuities in the evaluation of the psi functions, the derivatives are computed numerically using L.

The maximum likelihood for the parameters φ and λ are obtained by substituting the computed MLE values of φ and g into equations (51) and (52). Further substitution of the computed values of m, φ, λ and k into equations (37) and (38) yields the maximum likelihood estimates for the mean μ and standard deviation σ of y. Let these MLE estimators

$$\tilde{\theta} = \left[\tilde{\mu}, \tilde{\sigma}, \tilde{\lambda}, \tilde{k} \right].$$ (56)

Their sample variance-covariance matrix can be obtained from

$$\mathrm{var}\left(\tilde{\theta}\right) = Q\left(\tilde{\theta}\right)^{-1} = \sum_{t=1}^{T} \frac{\partial L_t\left(\tilde{\theta}\right)}{\partial \theta} \frac{\partial L_t\left(\tilde{\theta}\right)}{\partial \theta'}$$ (57)

or the inverse of the second-cross partial derivatives (Hessian matrix),

$$\mathrm{var}\left(\tilde{\theta}\right) = -H\left(\tilde{\theta}\right)^{-1} = -\sum_{t=1}^{T} \frac{\partial^2 L\left(\tilde{\theta}\right)}{\partial \theta \, \partial \theta'}.$$ (58)

Both of the above equations are asymptotically equivalent, because of the known MLE result

$$E\left(\sum_{t=1}^{T} \frac{\partial L_t}{\partial \theta} \frac{\partial L_t}{\partial \theta'} \right) = -E\left(\sum_{t=1}^{T} \frac{\partial^2 L_t}{\partial \theta \, \partial \theta'} \right)$$ (59)

Robust estimators for the variance-covariance matrix, which is more appropriate in case of mispecification, can be computed using the equation

$$\mathrm{var}\left(\theta\right) = -Q\left(\tilde{\theta}\right)^{-1} H\left(\tilde{\theta}\right) Q\left(\tilde{\theta}\right)^{-1}.$$ (60)

Note that all of the above partial derivatives are computed numerically.
Table 5 presents the estimates for the parameters of the SGED for

TABLE 5. SGED Estimates

	Daily	Weekly	Monthly	Quarterly
A. Boeing				
μ_a	12.8218	13.9143	14.1441	14.3197
	(2.78)**	(2.92)**	(2.87)**	(2.69)**
σ_a	31.0482	31.4127	31.1321	32.7129
	(116.74)**	(55.78)**	(27.67)**	(13.21)**
k	1.1246	1.2996	1.3575	1.3393
	(114.34)**	(41.7)**	(12.43)**	(5.81)**
λ	0.0347	–0.0053	–0.0858	–0.1618
	(7.31)**	(–0.26)	(–1.45)	(–1.18)
δ	0.0504	–0.008	–0.1291	–0.2413
SK	0.1245	–0.0156	–0.2352	–0.4448
KU	5.14	4.34	4.18	4.34
LogL	–20,605.2	–5,982.25	–1,721.1	–665.19
LR	1380.01**	143.8**	25.78**	7.69*
Observations	10,091	2,087	479	159
B. IBM				
μ_a	5.0837	4.5339	4.5133	4.914
	(1.34)	(1.13)	(1.19)	(0.93)
σ_a	25.6804	26.5179	24.8387	25.8231
	(115.2)**	(53.13)**	(28.27)**	(14.82)**
k	1.0525	1.0795	1.4139	1.4782
	(96.91)**	(37.04)**	(11.53)**	(4.45)**
λ	0.0077	0.0117	0.0432	–0.0174
	(1.2)	(1.28)	(1.18)	(–0.07)
δ	0.011	0.0169	0.0657	–0.0266
SK	0.0302	0.0447	0.1126	–0.0425
KU	5.59	5.41	3.99	3.81
LogL	–18,563.37	–5,564.74	–1,615.86	–630.59
LR	1,904.44**	334.39**	18.61**	4.89
Observations	10,091	2,087	479	159
C. DJIA				
μ_a	9.0953	8.2669	8.5374	8.1550
	(2.9)**	(3.16)**	(3.56)**	(3.11)**
σ_a	16.4907	16.9915	15.3405	15.9583
	(109.88)**	(51.34)**	(26.14)**	(13.93)**
k	0.9379	1.1378	1.2303	1.3822
	(147.94)**	(52.26)**	(13.9)**	(5.68)**
λ	0.0364	–0.0524	–0.1795	–0.2396
	(2.6)**	(–2.44)*	(–3.79)**	(–1.65)
δ	0.0506	–0.0762	–0.2625	–0.3556

(*Continued*)

TABLE 5.   (Continued)

	Daily	Weekly	Monthly	Quarterly
C. DJIA				
SK	0.1688	–0.1846	–0.5517	–0.6145
KU	6.61	5.08	4.81	4.33
$LogL$	–14,075.35	–4,654.71	–1,371.18	–549.95
LR	2,655.04**	348.16**	47.55**	14.07**
Observations	10,435	2,087	479	159
D. S&P100				
μ_a	10.1215	7.9407	7.9296	7.8124
	(1.2)	(2.51)*	(2.77)**	(2.63)**
σ_a	18.0505	17.6638	15.8855	16.6634
	(107.51)**	(45.06)**	(20.22)**	(9.64)**
k	0.8563	1.033	1.1542	1.1354
	(124.32)**	(43.18)**	(11.39)**	(4.1)**
λ	0.0319	–0.052	–0.2186	–0.2688
	(0.81)	(–4.55)**	(–4.95)**	(–4.33)**
δ	0.0432	–0.074	–0.3132	–0.3797
SK	0.1685	–0.2099	–0.7223	–0.887
KU	7.64	5.76	5.32	5.61
$LogL$	–11,330.51	–3,616.54	–1,062.64	–425.98
LR	2,504.08**	345.73**	50.5**	20.43**
Observations	8,042	1,608	369	123
E. S&P500				
μ_a	9.6742	8.4355	8.5264	8.3945
	(1.09)	(3.22)**	(3.52)**	(3.28)**
σ_a	16.7313	17.154	15.3151	16.2089
	(94.37)**	(52.4)**	(26.55)**	(13.3)**
k	0.9012	1.1012	1.2474	1.3726
	(73.25)**	(48.1)**	(12.69)**	(5.01)**
λ	0.029	–0.0668	–0.2028	–0.2998
	(0.93)	(–4.67)**	(–4.04)**	(–3.73)**
δ	0.0399	–0.0964	–0.2964	–0.4394
SK	0.1426	–0.2461	–0.6064	–0.7559
KU	7.02	5.31	4.79	4.49
$LogL$	–14,097.67	–4,659.94	–1,369.8	–550.54
LR	2,876.27**	357.91**	51.56**	16.71**
Observations	10,435	2,087	479	159
F. GBP/USD				
μ_a	–1.0656	–1.0781	–1.2184	–0.9716
	(–0.86)	(–0.68)	(–0.75)	(–0.56)
σ_a	9.6222	10.1237	10.3942	10.5671
	(99.17)**	(53.8)**	(27.57)**	(12.43)**

(*Continued*)

TABLE 5. (Continued)

	Daily	Weekly	Monthly	Quarterly
F. GBP/USD				
k	1.0037	1.1881	1.4258	1.2525
	(66.03)**	(31.92)**	(14.19)**	(5.89)**
λ	0.0025	–0.0603	–0.0436	–0.1139
	(0.15)	(–2.48)*	(–1.05)	(–0.64)
δ	0.0036	–0.0886	–0.0663	–0.1685
SK	0.0107	–0.1999	–0.1121	–0.3478
KU	5.97	4.83	3.96	4.6
Log$L$	–8,649.89	–3,588.95	–1,198.95	–484.86
LR	1,524.82**	195.51**	19.85**	10.82**
Observations	10,435	2,087	479	159
G. JY/USD				
μ_a	–1.9077	–1.803	–1.8054	–1.9732
	(–0.92)	(–0.87)	(–0.82)	(–0.82)
σ_a	11.1756	11.3946	11.4521	12.4452
	(93.48)**	(47.85)**	(22.74)**	(12.44)**
k	1.0207	1.2743	1.4376	1.6835
	(68.31)**	(32.13)**	(11.18)**	(4.57)**
λ	–0.005	–0.1182	–0.0664	–0.2529
	(–0.86)	(–4.58)**	(–0.81)	(–2.04)*
δ	–0.0071	–0.1754	–0.1011	–0.3879
SK	–0.0207	–0.352	–0.1686	–0.4933
KU	5.83	4.52	3.93	3.58
Log$L$	–7,422.2	–2,790.65	–902.15	–371.01
LR	1,244.32**	124.14**	11.09**	4.07
Observations	7,564	1,513	347	115

Note: The SGED parameters are obtained by maximizing the sample log-likelihood function with respect to the parameters μ, σ, k, and λ, where C, θ, and δ are given by equations (11)-(13), *sign* is the sign of the residual $(y_t - \mu + \delta\sigma)$, T is the sample size, ln is the natural logarithm. The statistics for the means and standard deviations are annualized using $\Delta t =$ 1/252, 1/52, 1/12,1/6, and 1/4 for the daily, weekly, monthly, and quarterly log-returns. The estimate for δ (Pearson's skewness) is computed by substituting the estimates for λ and k into equation (13). LR is the log-likelihood ratio statistics for testing the null-hypothesis that the series follow the normal distribution against the alternative hypothesis of the SGED distribution. The LR statistic follows the $\chi^2(2)$ with two degrees of freedom. *,** imply statistically significant the five- and one-percent level of significance, respectively.

all log-return series and frequencies. The first eight rows report the estimates and *t*-values for the mean, standard deviation, kurtosis, and skewness parameters of the SGED. The *t*-values for these estimates are enclosed in parentheses below the estimates. As in table 2, the estimated means and standard deviations of log-returns are annualized using the
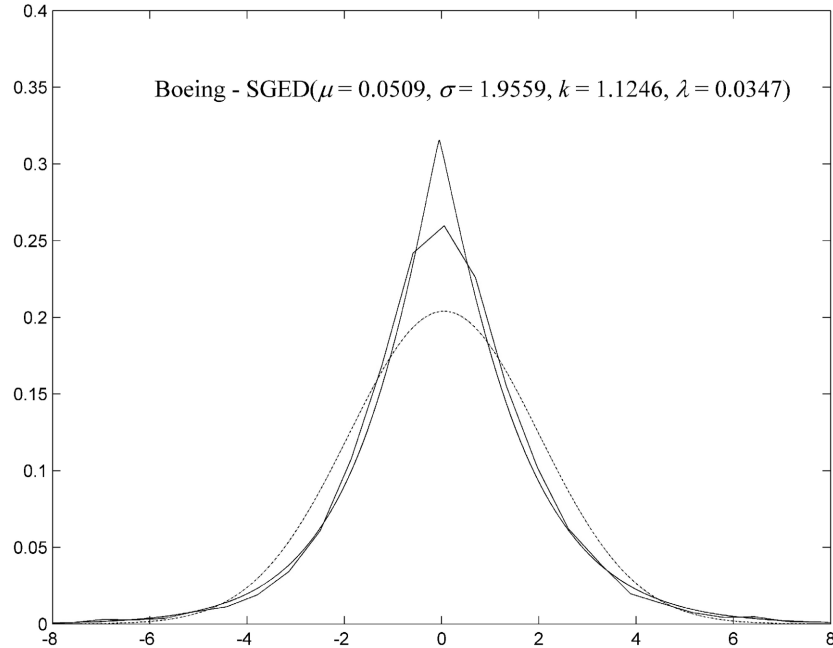
FIGURE 3.— Boeing, Empirical Distributions of Log-Returns

formulas $\tilde{\mu}_a = \tilde{\mu}/\Delta t$ and $\tilde{\sigma}_a = \tilde{\sigma}/\sqrt{\Delta t}$, where $\Delta t = 1/252$, 1/52, 1/12 and
1/4 for daily, weekly, monthly and quarterly log-returns, respectively.
The remaining rows contain estimates of Pearson's skewness δ,
skewness SK and kurtosis KU calculated using equations (34), (35) and
(37), the sample log-likelihood function (Log$L$), and the log-likelihood
ratio (LR) for testing the null hypothesis of normality against the
alternative hypothesis of the SGED.

    The estimated means and standard deviations of each series are
similar across the four data frequencies and close to the respective
estimates in table 2. The estimates of the kurtosis parameter k ranges
between 0.8563 and 1.1246 for daily, 1.033 and 1.2996 for weekly,
1.1542 and 1.4376 for monthly and 1.1354 and 1.6835 for quarterly log-
returns. Its respective means across the four frequencies are 0.9853,
1.1591, 1.3238 and 1.3777. These numbers indicate that, on average,
daily log-returns follow the Laplace distribution. The other log-return
data frequencies diverge from the Laplace distribution but do not
become normal ($k = 2$).

IBM - SGED($\mu = 0.0202$, $\sigma = 1.6177$, $k = 1.0525$, $\lambda = 0.0077$)
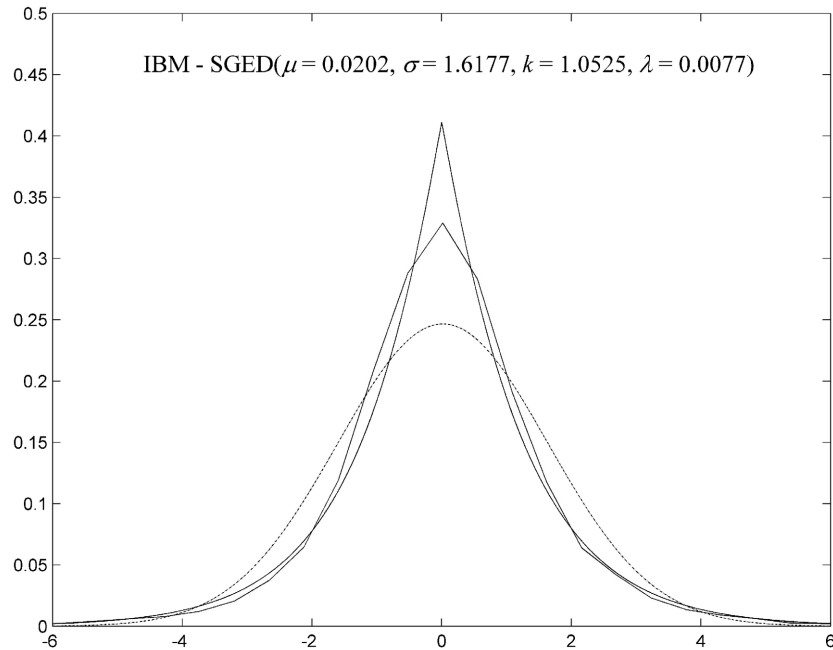
FIGURE 4.— IBM, Empirical Distributions of Log-Returns

The kurtosis *KU* values range between 5.14 and 7.64 for daily, 4.34 and 5.76 for weekly, 3.93 and 5.32 for monthly and 3.58 and 5.61 for quarterly log-returns. Their respective means across the four frequencies are 6.26, 5.03, 4.43 and 4.39. These numbers also indicate that, on average, daily log-returns have a kurtosis value of 6.26 close to that Laplace distribution ($KU = 6$). The kurtosis values of the remaining log-return frequencies are smaller, but do not approach the kurtosis of the normal distribution ($KU = 3$).

The results for skewness (λ) indicate, in general, the presence of negative skewness in the data. Twenty out of the twenty-eight estimated values of λ are negative. In eleven of these cases the estimates are statistically significant. In eight cases the estimates of λ are positive but only in two of them are statistically significant. The results for skewness are consistent with those of table 2.

Because the normal distribution is nested under the SGED, a log-likelihood ratio statistic could be used to test the null hypothesis that log-returns are normally distributed against the alternative hypothesis

$$\text{DJIA - SGED}(\mu = 0.0348,\ \sigma = 1.0207,\ k = 0.9379,\ \lambda = 0.0364)$$
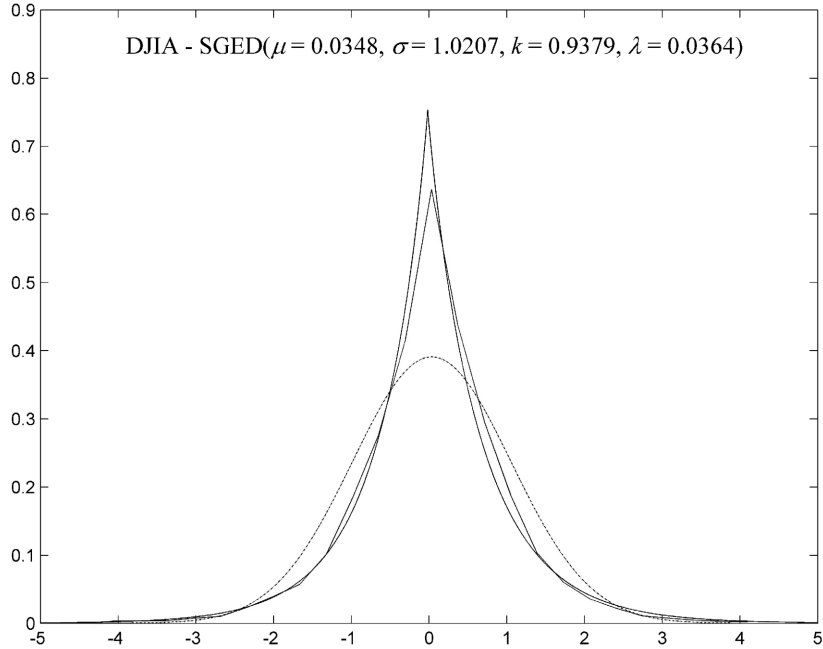
FIGURE 5.— DJIA, Empirical Distributions of Log-Returns

that they are SGED distributed. The null hypothesis for testing normality is H_0: $k = 2$ and $\lambda = 0$ and the alternative hypothesis is $H_1 : k \in R^+$ and $|\lambda| \leq 1$. The statistic is computed using the formula $LR = -2 \times (\text{Log}L_0 - \text{Log}L)$, where $\text{Log}L_0$ is the maximum value of the log-likelihood function under the null hypothesis of normality and $\text{Log}L$ is the maximum value of the log-likelihood function under the SGED specification. The LR statistic is asymptotically distributed as a chi-square with two degrees of freedom, $\chi^2(2)$. With one exception (quarterly log-returns for IBM), the LR statistics reject the null hypothesis of normality. Note that for a small sample the statistical power of the LR statistic is weak and may accept the null hypothesis of normality when it should be rejected.

Figures 3 to 9 present the empirical distributions for the daily log-returns of the seven series. The figures for weekly, monthly and quarterly log-returns are qualitatively similar, therefore, are omitted. The estimated SGED distributions are represented by the solid curves and the estimated non-parametric distributions by the piecewise linear

$$\text{S\&P 100 - SGED}(\mu = 0.0388,\ \sigma = 1.1173,\ k = 0.8563,\ \lambda = 0.0319)$$
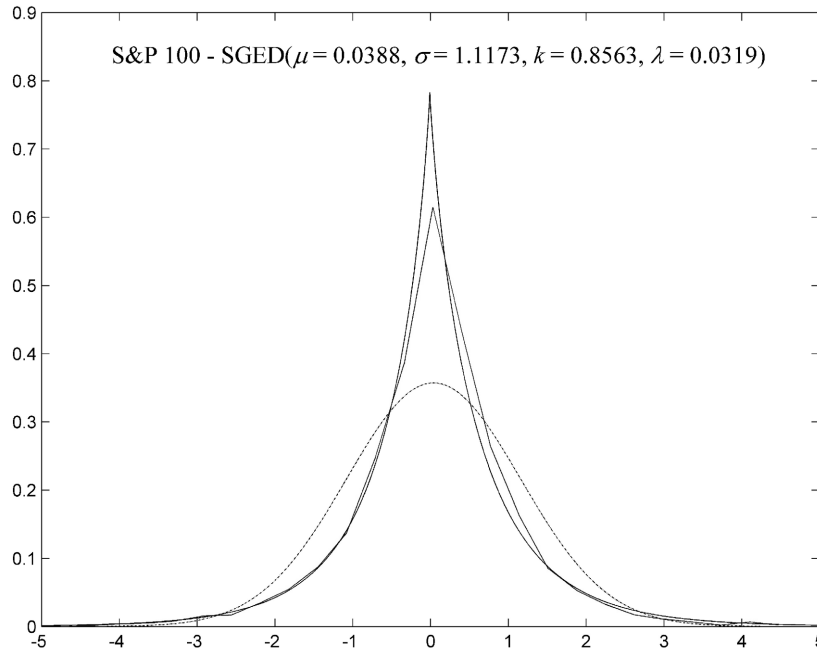
FIGURE 6.— S&P 100, Empirical Distributions of Log-Returns

curves; see Tapia and Thompson (1978), section 2.5 for the latter. For comparison purposes, the estimated normal distributions are also presented by the dotted curves. These graphs demonstrate that the SGED provides a very good fit to the empirical distribution of log-returns of all series. In some cases, the estimated SGED and the non-parametric probability curves are indistinguishable. Moreover, they deviate significantly from those of the normal distribution.

VI. Call Option Pricing

This section derives the SGED option pricing model and demonstrates its relationship to several well-known pricing models for European call options. These are the Black and Scholes' (1973) model, Merton's (1973) continuous dividend yield model, Biger and Hull's (1983) currency model, and Black's (1976) currency futures model.

$$\text{S\&P 500 - SGED}(\mu = 0.0371,\ \sigma = 1.0356,\ k = 0.9012,\ \lambda = 0.029)$$
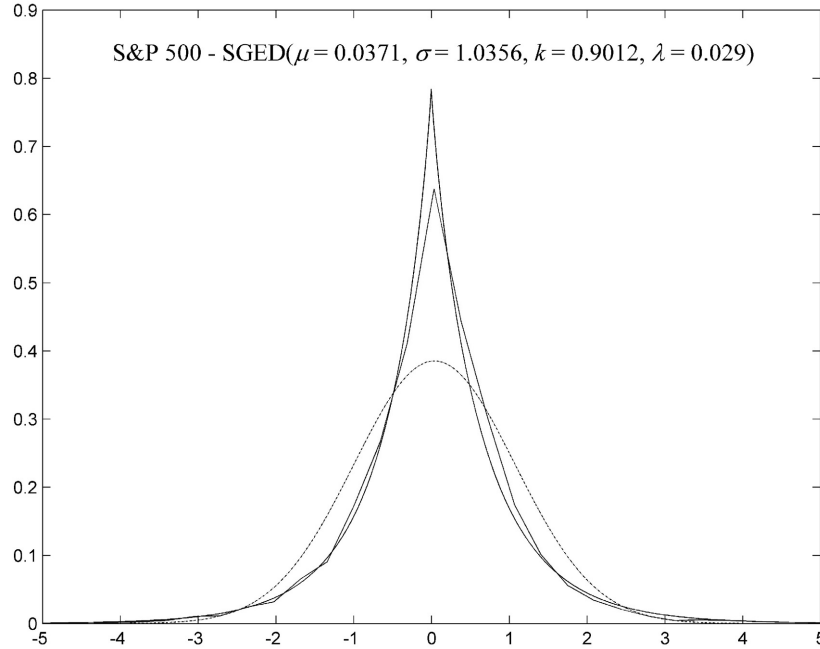
FIGURE 7.— S&P 500, Empirical Distributions of Log-Returns

A. SGED Option Pricing Model

Consider a European call option with a strike price X written on an asset with a current (spot) price P_0 that pays no dividends and expires n periods from time $t = 0$. Let $\tau = n\,\Delta t$ be the option's time to expiration. The value of this option at expiration is equal to $C_n = \max(P_n - X, 0)$, where P_n is the price of the underlying asset at expiration. Observe that

$$P_n = P_0 \exp\left(\ln\left(P_n/P_0\right)\right) = P_0\, e^{Y_n}, \tag{61}$$

where

$$Y_n = \ln\left(P_n/P_0\right) = \mu_a \tau + z\,\sigma_a\sqrt{\tau} \tag{62}$$

is the log-return to prevail over the life of the option as a function of the mean and the standard deviation of Y_n and the standardized random variable z with probability mass function given by $dF(z) = f(z)dz$.
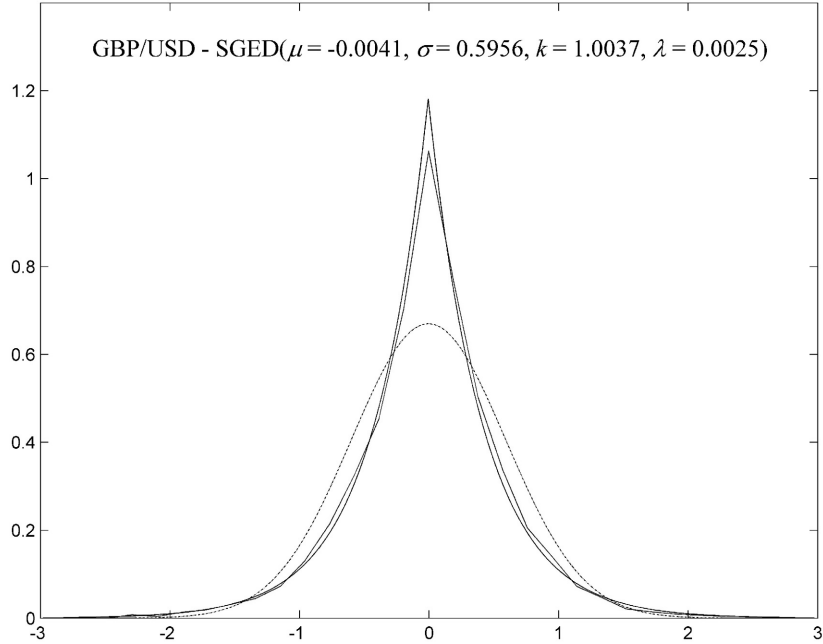
FIGURE 8.— GBP/USD, Empirical Distributions of Log-Returns

The expected value of the option at time n is

$$E(C_n) = E\big(\max(P_n - X, 0)\big)$$

$$= P_0 E\Big(\max\big(e^{Y_n} - X/P_0, 0\big)\Big). \tag{63}$$

The presence of a positive pay-off at expiration (i.e., $P_n > X$) implies that

$$Y_n = \mu_a \tau + z\sigma_a\sqrt{\tau} > -\ln(P_0/X). \tag{64}$$

or

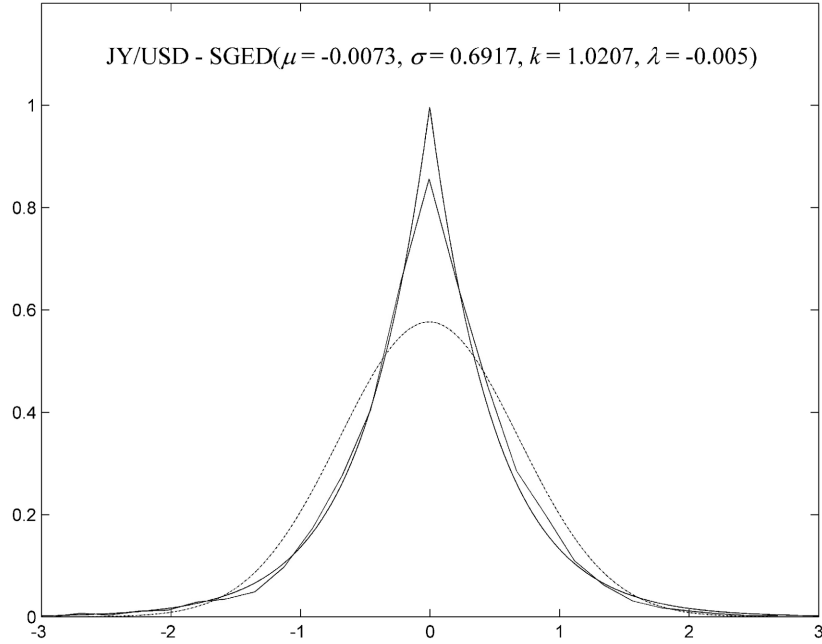$$z > -\frac{\ln(P_0/X) + \mu_a\tau}{\sigma_a\sqrt{\tau}} \equiv h .$$

The function $E(C_n)$ can be expressed in terms of z

FIGURE 9.— JY/USD, Empirical Distributions of Log-Returns

$$E\left(C_n\right) = P_0 \int_h^\infty \left(e^{\mu_a \tau + z\,\sigma_a \sqrt{\tau}} - X/P_0 \right) dF\left(z\right)$$

$$= P_0 e^{\mu_a \tau} \int_h^\infty e^{z\,\sigma_a \sqrt{\tau}} dF_z\left(z\right) - X \int_h^\infty dF_z\left(z\right)$$

$$= P_0 e^{\mu_a \tau} \int_h^\infty e^{z\,\sigma_a \sqrt{\tau}} dF_z\left(z\right) - X\left(1 - F_z\left(h\right)\right)$$

where h is as defined above.

The option price at $t = 0$ is equal to present value of $E(C_n)$. That is,

$$C_0 = e^{-r\tau} E\left(C_n\right)$$

$$= P_0 e^{(\mu_a - r)\tau} \int_h^\infty e^{z\sigma_a\sqrt{\tau}} dF_z(z) - e^{-r\tau} X\left(1 - F_z(h)\right) \tag{65}$$

where r is an equilibrium discount rate of return for buyers and issuers of call options. Under risk neutrality, this discount rate is taken to be equal to the risk free rate. In the above equation, the parameters k, and λ could be obtained by maximizing the SGED sample log-likelihood function, equation 50, using data that best reflect the distribution of log-returns of the underlying asset during the life of the option. As for σ_α measures of implied volatility may be used. The values of P_0 and X are always known and do not change over the life of the call option.

B. Special Cases of SGED Option Pricing Model

Under normality, the integral in equation (36), see appendix B, becomes

$$\int_h^\infty e^{z\sigma_a\sqrt{\tau}} f(z) dz = e^{\frac{1}{2}\sigma_a^2\tau} \Phi\left(-h + \sigma_a\sqrt{\tau}\right) = e^{\frac{1}{2}\sigma_a^2\tau} N(d_1) \tag{66}$$

where

$$d_1 \equiv -h + \sigma_a\sqrt{\tau} = \frac{\ln(P_0/X) + \left(\mu_a + \sigma_a^2\right)\tau}{\sigma_a\sqrt{\tau}}. \tag{67}$$

Moreover, the probability,

$$1 - F_z(h) = 1 - \Phi(h) = \Phi(-h) = N(d_2), \tag{68}$$

where

$$d_2 = d_1 - \sigma_a\sqrt{\tau}. \tag{69}$$

Note that d_1 and d_2 is standard notation used in option pricing models and N is used insetad of Φ. Substitution of these quantities into the option pricing equation of equation (65) yields

$$C_0 = P_0 e^{\left(\mu_a + \frac{1}{2}\sigma_a^2 - r\right)\tau} N(d_1) - e^{-r\tau} X N(d_2), \tag{70}$$

which can be used as pricing kernel in cases where log-returns are

normal or returns are log-normal. In the above equation, the quantity $\mu_a + \frac{1}{2}\sigma_2^2$ measures the stock's expected return; see equation (27).

Black and Scholes Model

In the absence of arbitrage opportunities, the expected annual return of a stock paying no dividend is equal to the risk-free rate. That is,

$$\mu_a + \tfrac{1}{2}\sigma_a^2 = r . \qquad (71)$$

Substitution of this equality into equation (70) gives the Black and Scholes (1973) option pricing model,

$$C_0 = P_0\, N(d_1) - e^{-r\tau} X\, N(d_2), \qquad (72)$$

where

$$d_1 = \frac{\ln(P_0/X) + \left(r + \tfrac{1}{2}\sigma_a^2\right)\tau}{\sigma_a\sqrt{\tau}} \qquad (73)$$

and $d_2 = d_1 - \sigma_a\sqrt{\tau}.$

Merton's Model

In the case of a stock paying continuous dividend at an annual rate q, the expected stock return, including dividends, is equal to the risk free borrowing rate r. That is,

$$\mu_a + \tfrac{1}{2}\sigma_a^2 + q = r. \qquad (74)$$

Substitution of this equality into equation (70) gives Merton's (1973) continuous dividend yield model,

$$C_0 = P_0\, e^{-q\tau} N(d_1) - e^{-r\tau} X\, N(d_2), \qquad (75)$$

where

$$d_1 = \frac{\ln(P_0/X) + \left(r - q + \tfrac{1}{2}\sigma_a^2\right)\tau}{\sigma_a\sqrt{\tau}} \qquad (76)$$

and $d_2 = d_1 - \sigma_a\sqrt{\tau}.$

Currency Option Pricing Model

In the absence of arbitrage opportunities, the expected growth rate of a currency is equal to the difference between the domestic r and foreign interest rates r_f. That is,

$$\mu_a + \tfrac{1}{2}\sigma_a^2 = r - r_f. \tag{77}$$

Substitution of this equality into equation (70) gives Biger and Hull's (1983) currency option pricing model,

$$C_0 = P_0\, e^{-r_f \tau} N(d_1) - e^{-r\tau} X\, N(d_2), \tag{78}$$

where

$$d_1 = \frac{\ln(P_0/X) + \left(r - r_f + \tfrac{1}{2}\sigma_a^2\right)\tau}{\sigma_a \sqrt{\tau}} \tag{79}$$

and $d_2 = d_1 - \sigma_a \sqrt{\tau}$.

Currency Future Option Model

Another variation of the model above is obtained by substituting the exchange rate parity

$$F_n = P_0 e^{(r - r_f)\tau}, \tag{80}$$

where F_n is the forward exchange rate into equation (70). That is,

$$C_0 = e^{-r\tau}\left[F_n N(d_1) - X\, N(d_2)\right], \tag{81}$$

where

$$d_1 = \frac{\ln(F_n/X) + \tfrac{1}{2}\sigma_a^2\,\tau}{\sigma_a \sqrt{\tau}} \tag{82}$$

and $d_2 = d_1 - \sigma_a \sqrt{\tau}$.

This formulation is similar to Black's (1976) model for pricing options on currency futures, where F_n represents the futures price.

General Comments

Note that the four variation of the option pricing model presented above

are based on the assumptions of normality of log-returns and absence of arbitrage opportunities or that the mean return for stocks and stock indices is equal to the risk free rate and for currencies equal to the spread between the domestic and foreign interest rates. Table 5 presents estimates for the mean returns of the seven series. The means of all stocks and stock indices are considerably higher from the U.S. t-bill rate, which has ranged between 3 and 6 percent during the past ten years. The mean returns, however, for the two currencies are close to the spread between the t-bill rate in the U.S. and those in the U.K. and Japan. These differences are indirectly compensated with the use of implied volatility measures which are typically larger than those obtained through estimation using historical data.

VII. Summary and Concluding Remarks

Analytical formulas and test statistics developed in this article established that the presence of skewness and kurtosis in financial data was the result of higher-order moment dependencies, such as conditional heteroskedasticity, asymmetric volatility, and other non-linear dependencies, that existed mainly in daily and weekly log-returns. These dependencies prevented monthly, bimonthly, and quarterly log-returns to obey the normality law implied by the central limit theorem. As a consequence, prices of these assets violated the geometric Brownian motion often assumed in pricing options and other derivative assets.

The above findings necessitated the development of distributions that provided better approximations to the empirical distributions of log-returns. This article developed a skewed extension of the generalized error distribution (SGED) and assessed its ability to fit the empirical distribution of log-returns of several popular financial assets at various data frequencies. The results showed that the SGED provided a good fit to the empirical distribution of the data.

A model for pricing European call options based on the SGED was developed that accommodated skewness and kurtosis in the data and relaxed the assumptions of risk neutrality and no arbitrage opportunities. Nevertheless, under the standard assumptions in option pricing of normality of log-returns, risk neutrality, and absence of arbitrage opportunities, the SGED model produced as special cases several well-known option pricing models, such as the Black and Scholes (1973) model, Merton's (1973) continuous dividend yield model, Biger and

Hull's (1983) currency model, and Black's (1976) currency futures model.

Appendix A. SGED's Moments

Consider the non-centered SGED probability density function

$$f(u) = C \exp\left(-\frac{1}{k}\frac{|u|^k}{\left(1 + sgn(u)\lambda\right)^k \varphi^k}\right), \tag{A1}$$

where C is a scaling constant. The rth non-centered moment of u is

$$M_{u,r} = Eu^r = C(-1)^r \int_0^\infty u^r \exp\left(-\frac{1}{k}\frac{u^k}{\left(1-\lambda\right)^k \varphi^k}\right) du$$

$$+ C\int_0^\infty u^r \exp\left(-\frac{1}{k}\frac{u^k}{\left(1+\lambda\right)^k \varphi^k}\right) du \tag{A2}$$

Define

$$t = \frac{1}{k}\frac{u^k}{\left(1\pm\lambda\right)^k \varphi^k} \tag{A3}$$

and note that

$$u = \left(1\pm\lambda\right)\varphi k^{\frac{1}{k}} t^{\frac{1}{k}} \tag{A4}$$

and

$$du = \left(1\pm\lambda\right)\varphi k^{\frac{1}{k}-1} t^{\frac{1}{k}-1} dt . \tag{A5}$$

Substitution of (A3) – (A5) into (A2) gives

$$M_{u,r} = C(-1)^r k^{\frac{r+1}{k}-1} \left(1-\lambda\right)^{r+1} \varphi^{r+1} \int_0^\infty t^{\frac{r+1}{k}-1} e^{-t} dt$$

$$+ Ck^{\frac{r+1}{k}-1}\left(1+\lambda\right)^{r+1} \varphi^{r+1} \int_0^\infty t^{\frac{r+1}{k}-1} e^{-t} dt$$

$$= C k^{\frac{r+1}{k}-1} \left[(-1)^r (1-\lambda)^{r+1} + (1+\lambda)^{r+1} \right] \varphi^{r+1} \Gamma\left(\frac{r+1}{k}\right) \qquad \text{(A6)}$$

Setting $M_{u,0} = 1$, yields

$$M_{u,0} = C k^{\frac{1}{k}-1} \left[(1-\lambda) + (1+\lambda) \right] \varphi \Gamma\left(\frac{1}{k}\right) = 2C k^{\frac{1}{k}-1} \varphi \Gamma\left(\frac{1}{k}\right) = 1,$$

thus

$$C = \frac{k^{1-\frac{1}{k}}}{2\varphi} \Gamma\left(\frac{1}{k}\right)^{-1}. \qquad \text{(A7)}$$

Substitution of (A7) into (A1) and (A6) gives

$$f(u) = \frac{k^{1-\frac{1}{k}}}{2\varphi} \Gamma\left(\frac{1}{k}\right)^{-1} \exp\left(-\frac{1}{k} \frac{|u|^k}{\left(1 + sgn(u)\lambda\right)^k \varphi^k} \right) \qquad \text{(A8)}$$

and

$$M_{u,r} = \frac{1}{2} \left[(-1)^r (1-\lambda)^{r+1} + (1+\lambda)^{r+1} \right] G_r \varphi^r = A_r \varphi^r \qquad \text{(A9)}$$

where

$$G_r = k^{\frac{r}{k}} \Gamma\left(\frac{r+1}{k}\right) \Gamma\left(\frac{1}{k}\right)^{-1}. \qquad \text{(A10)}$$

and

$$A_r = \frac{1}{2} \left[(-1)^r (1-\lambda)^{r+1} + (1+\lambda)^{r+1} \right] G_r. \qquad \text{(A11)}$$

The expected value of u is

$$Eu = M_{u,1} = A_1 \varphi = 2\lambda G_1 \varphi \qquad \text{(A12)}$$

where

$$G_1 = k^{\frac{1}{k}} \Gamma\left(\frac{2}{k}\right) \Gamma\left(\frac{1}{k}\right)^{-1}. \qquad \text{(A13)}$$

The variance of u

$$m_2 = \mathrm{var}(u) = Eu^2 - (Eu)^2 = \left(A_2 - A_1^2\right)\varphi^2 \qquad \text{(A14)}$$

where

$$A_2 = \frac{1}{2}\left[(-1)^2(1-\lambda)^3 + (1+\lambda)^3\right]G_2 = \left(1+3\lambda^2\right)G_2 \qquad \text{(A15)}$$

and

$$G_2 = k^{\frac{2}{k}}\Gamma\left(\frac{3}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1}. \qquad \text{(A16)}$$

The third of u is

$$M_{u,3} = A_3\varphi^3 = \frac{1}{2}\left[(-1)^3(1-\lambda)^4 + (1+\lambda)^4\right]G_3\varphi^3$$

$$= 2\lambda\left(1+\lambda^2\right)G_3\varphi^3 \qquad \text{(A17)}$$

where

$$G_3 = k^{\frac{3}{k}}\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1}. \qquad \text{(A18)}$$

The fourth moment is

$$M_{u,4} = A_4\varphi^4 = \frac{1}{2}\left[(-1)^4(1-\lambda)^5 + (1+\lambda)^5\right]G_4\varphi^4$$

$$= \left(1+10\lambda^2 + 5\lambda^4\right)G_4\varphi^4 \qquad \text{(A19)}$$

where

$$G_4 = k^{\frac{4}{k}}\Gamma\left(\frac{5}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1}. \qquad \text{(A20)}$$

The third moment of u is

$$m_3 = E(u-Eu)^3 = Eu^3 - 3Eu^2Eu + 3Eu(Eu)^2 - (Eu)^3$$

$$= \left(A_3 - 3A_1A_2 + 2A_1^3\right)\varphi^3 \qquad \text{(A21)}$$

The fourth centered moment of u is

$$m_4 = E(u - Eu)^4$$

$$= Eu^4 - 4Eu^3Eu + 6Eu^2(Eu)^2 - 4Eu(Eu)^3 + (Eu)^4$$

$$= \left(A_4 - 4A_3A_1 + 6A_2A_1^2 - 3A_1^4\right)\varphi^4 . \qquad (A22)$$

The standardized skewness and kurtosis are

$$SK = \frac{m_3}{m_2^{3/2}} = \frac{A_3 - 3A_1A_2 + 2A_1^3}{\left(A_2 - A_1^2\right)^{3/2}} \qquad (A23)$$

and

$$KU = \frac{m_4}{m_2^2} = \frac{A_4 - 4A_3A_1 + 6A_2A_1^2 - 3A_1^4}{\left(A_2 - A_1^2\right)^2} . \qquad (A24)$$

Appendix B. Integral Evaluation Under Normality

Under normality, the integral

$$\int_h^\infty e^{z\sigma_a\sqrt{\tau}} f(z)\,dz = \int_h^\infty e^{z\sigma_a\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz$$

$$= e^{\frac{1}{2}\sigma_a^2\tau} \int_\infty^{-h} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z - \sigma_a\sqrt{\tau}\right)^2}\,dz$$

$$= e^{\frac{1}{2}\sigma_a^2\tau} \int_\infty^{-h+\sigma_a\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2}\,ds = e^{\frac{1}{2}\sigma_a^2\tau}\Phi\left(-h + \sigma_a\sqrt{\tau}\right) \qquad (B1)$$

where $s = z - \sigma_a\sqrt{\tau}$, $dz = ds$ and Φ is the cumulative density for the standard normal distribution.

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